Parameter estimation (cont.)

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Outline

- Normal model, unknown mean
  - Jeffreys prior
  - Natural conjugate prior
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- Normal model, unknown variance
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Jeffreys prior for μ

Theorem

If $Y_i \stackrel{iid}{\sim} N(\mu, s^2)$ (s^2 known), Jeffreys prior for μ is $p(\mu) \propto 1$.

Proof.

Since the normal distribution with unknown mean is an exponential family, use Casella & Berger Lemma 7.3.11

$$
\begin{aligned}
-E_y\left[\frac{\partial^2}{\partial\mu^2}\log p(y|\mu)\right] &= -E_y\left[\frac{\partial^2}{\partial\mu^2}\left(-\log(2\pi s^2)/2 - \frac{1}{2s^2}\sum_{i=1}^n (y_i-\mu)^2\right)\right] \\
&= -E_y\left[\frac{\partial^2}{\partial\mu^2}\left(-\log(2\pi s^2)/2 - \frac{1}{2s^2}\left(\sum_{i=1}^n y_i^2 - 2\mu n\overline{y} + n\mu^2\right)\right)\right] \\
&= -E_y\left[\frac{\partial}{\partial\mu}\left(-\frac{1}{2s^2}(-2n\overline{y} + 2n\mu)\right)\right] \\
&= -E_y\left[-\frac{1}{2s^2}(2n)\right] \\
&= n/s^2 \\
p(\mu) &\propto \sqrt{|\mathcal{I}(\mu)|} = \sqrt{n/s^2} \\
&\propto 1
\end{aligned}
$$

So Jeffreys prior for μ is $p(\mu) \propto 1$. □

Posterior propriety

Since $\int_{-\infty}^{\infty} 1 d\mu$ is not finite, we need to check posterior propriety.

Theorem

For $n > 0$, the posterior for a normal mean (known variance) using Jeffreys prior is proper.

Proof.

The posterior is

$$\begin{aligned} p(\mu|y) &\propto p(y|\mu)p(\mu) \\ &\propto \exp\left(-\frac{1}{2s^2}\sum_{i=1}^n (y_i-\mu)^2\right) \times 1 \\ &\propto \exp\left(-\frac{1}{2s^2}\left[-2\mu n\overline{y} + n\mu^2\right]\right) \\ &= \exp\left(-\frac{1}{2s^2/n}\left[\mu^2 - 2\mu\overline{y}\right]\right). \end{aligned}$$

This is the kernel of a normal distribution with mean $\overline{y}$ and variance s^2/n which is proper if $n > 0$. $\square$

Natural conjugate prior

Let $Y_i \stackrel{iid}{\sim} N(\mu, s^2)$ with s^2 known. The likelihood is

$$\begin{aligned} L(\mu) &= \exp\left(-\frac{1}{2s^2/n}\left[\mu^2 - 2\mu\overline{y}\right]\right) \\ &\propto \exp\left(-\frac{1}{2}\left[\frac{n}{s^2}\mu^2 - 2\mu\frac{n}{s^2}\overline{y}\right]\right) \end{aligned}$$

This is the kernel of a normal distribution, so the natural conjugate prior is $\mu \sim N(m, C)$.

$$\begin{aligned} p(\mu|y) &\propto p(y|\mu)p(\mu) = L(\mu)p(\mu) \\ &= \exp\left(-\frac{1}{2}\left[\frac{n}{s^2}\mu^2 - 2\mu\frac{n}{s^2}\overline{y}\right]\right)\exp\left(-\frac{1}{2}\left[\frac{1}{C}\mu^2 - 2\mu\frac{1}{C}m\right]\right) \\ &= \exp\left(-\frac{1}{2}\left[\left(\frac{1}{C} + \frac{n}{s^2}\right)\mu^2 - 2\mu\left(\frac{1}{C}m + \frac{n}{s^2}\overline{y}\right)\right]\right) \\ &= \exp\left(-\frac{1}{2\left(\frac{1}{C}+\frac{n}{s^2}\right)^{-1}}\left[\mu^2 - 2\mu\frac{1}{\left(\frac{1}{C}+\frac{n}{s^2}\right)}\left(\frac{1}{C}m + \frac{n}{s^2}\overline{y}\right)\right]\right) \end{aligned}$$

This is the kernel of a $N(m', C')$ where

$$C' = \left[C^{-1} + n/s^2\right]^{-1} \qquad m' = C'\left[C^{-1}m + n/s^2\overline{y}\right]$$

Normal mean posterior comments

Let $P = 1/C$, $P' = 1/C'$, and $Q = 1/s^2$ be the relevant precisions (inverse variances), then

- The posterior precision is the sum of the prior and observation precisions.

$$P' = P + \sum_{i=1}^{n} Q = P + nQ.$$

- The posterior mean is a precision weighted average of the prior and data.

$$\begin{aligned} m' &= \frac{1}{P'}\left[Pm + nQ\overline{y}\right] \\ &= \frac{P}{P'}m + n\frac{Q}{P'}\overline{y} \\ &= \frac{P}{P'}m + \sum_{i=1}^{n} \frac{Q}{P'}y_i \end{aligned}$$

- Jeffreys prior/posterior are the limits of the conjugate prior/posterior as $C \to \infty$, i.e.

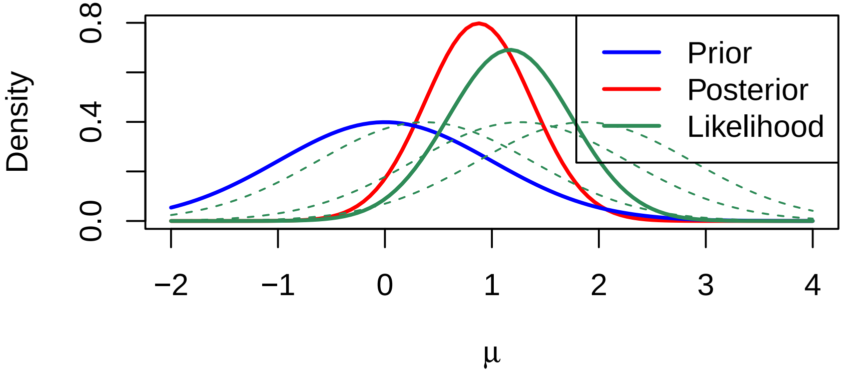
$$\lim_{C\to\infty} N(m, C) \stackrel{d}{\to} \propto 1 \qquad \lim_{C\to\infty} N(m', C') \stackrel{d}{\to} N(\overline{y}, s^2/n)$$

Example

Consider $Y_i \stackrel{ind}{\sim} N(\mu, 1)$ and $\mu \sim N(0, 1)$.

```
# Prior
m = 0
C = 1; P = 1/C

# Data
mu = 1
s2 = 1; Q = 1/s2
n = 3
set.seed(6); (y = rnorm(n,mu,sqrt(1/Q)))

[1] 1.2696060 0.3700146 1.8686598

# Posterior
nQ = n*Q
Pp = P+nQ
mp = (P*m+nQ*mean(y))/Pp
```

Normal model with unknown mean, normal prior



Theorem

If $Y_i \stackrel{iid}{\sim} N(m,\sigma^2)$ (m known), Jeffreys prior for σ^2 is $p(\sigma^2) \propto 1/\sigma^2$.

Proof.

Since the normal distribution with unknown variance is an exponential family, use Casella & Berger Lemma 7.3.11.

$$\begin{aligned}
-E_y\left[\frac{\partial^2}{\partial(\sigma^2)^2}\log p(y|\sigma^2)\right] &= -E_y\left[\frac{\partial^2}{\partial(\sigma^2)^2} - n\log(2\pi\sigma^2)/2 - \frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-m)^2\right] \\
&= -E_y\left[\frac{\partial}{\partial(\sigma^2)} - \frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2}\sum_{i=1}^n (y_i-m)^2\right] \\
&= -E_y\left[\frac{n}{2(\sigma^2)^2} - \frac{1}{(\sigma^2)^3}\sum_{i=1}^n (y_i-m)^2\right] \\
&= -\frac{n}{2(\sigma^2)^2} + \frac{n}{(\sigma^2)^3}\sigma^2 \\
&= \frac{n}{2}(\sigma^2)^{-2} \\
p(\sigma^2) &\propto \sqrt{|\mathcal{I}(\sigma^2)|} \propto 1/\sigma^2
\end{aligned}$$

So Jeffreys prior is $p(\sigma^2) \propto 1/\sigma^2$. □

Posterior propriety

Since $\int_0^\infty 1/\sigma^2 d\sigma^2$ is not finite, we need to check posterior propriety.

Theorem

For $n > 0$ and at least one $y_i \ne m$, the posterior for a normal variance (known mean) using Jeffreys prior is proper.

Proof.

The posterior is

$$\begin{array}{rl} p(\sigma^2|y) & \propto p(y|\sigma^2)p(\sigma^2) \\ & = (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n [y_i - m]^2\right)(\sigma^2)^{-1} \\ & \propto (\sigma^2)^{-n/2-1} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n [y_i - m]^2\right) \end{array}$$

This is the kernel of an inverse gamma distribution with shape $n/2$ and scale $\sum_{i=1}^n [y_i - m]^2/2$ which will be proper so long as $n > 0$ and at least one $y_i \ne m$. $\square$

Natural conjugate prior

Let $Y_i \stackrel{iid}{\sim} N(m, \sigma^2)$ with m known. The likelihood is

$$L(\sigma^2) \quad \propto (\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^n [y_i - m]^2\right)$$

This is the kernel of an inverse gamma distribution, so the natural conjugate prior is $IG(a, b)$.

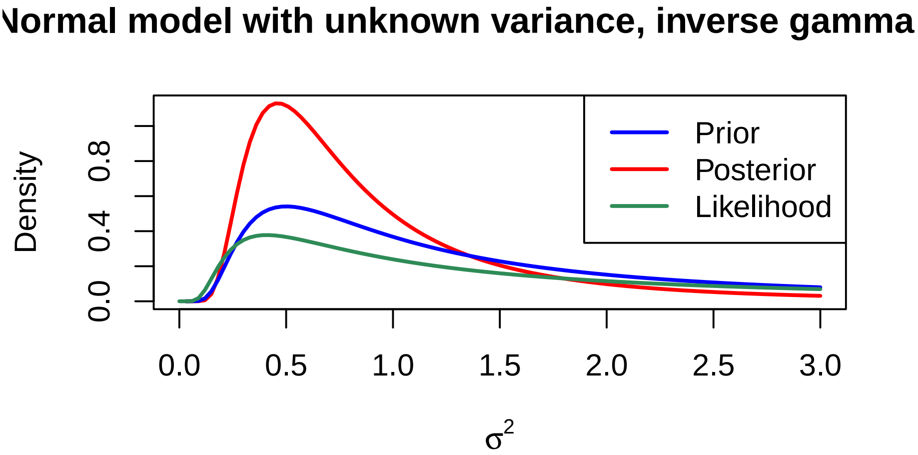
$$\begin{array}{rl} p(\sigma^2|y) & \propto p(y|\sigma^2)p(\sigma^2) \\ & = (\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^n [y_i - m]^2\right) (\sigma^2)^{-a-1} \exp(-b/\sigma^2) \\ & = (\sigma^2)^{-(a+n/2)-1} \exp\left(-\frac{1}{\sigma^2}\left[b + \sum_{i=1}^n [y_i - m]^2/2\right]\right) \end{array}$$

This is the kernel of an inverse gamma distribution with shape $a + n/2$ and scale $b + \sum_{i=1}^n [y_i - m]^2/2$.

Example

Suppose $Y_i \stackrel{ind}{\sim} N(1, \sigma^2)$ and $\sigma^2 \sim IG(1, 1)$.

```
# Prior
a = b = 1

# Data
m = 1
n = length(y)
y

[1] 1.2696060 0.3700146 1.8686598

# Posterior
ap = a + n/2
bp = b + sum((y-m)^2)/2

ap

[1] 2.5

bp

[1] 1.612069
```


Normal model with unknown variance, inverse gamma
Density
0.0
0.4
0.8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
σ2
Prior
Posterior
Likelihood

Summary

Suppose $Y_i \sim N(\mu, \sigma^2)$.

- μ unknown (σ^2 known)
  - Jeffreys prior: $p(\mu) \propto 1$ (think of this as $N(0, \infty)$)
  - Natural conjugate prior: $N(m, C)$
  - Posterior $N(m', C')$ with
    - $C' = [1/C + n\sigma^{-2}]^{-1}$
    - $m' = C'[m/C + n\sigma^{-2}\overline{y}]$
- σ^2 unknown (μ known)
  - Jeffreys prior: $p(\sigma^2) \propto 1/\sigma^2$ (think of this as $IG(0, 0)$)
  - Natural conjugate prior $IG(a, b)$
  - Posterior $IG\left(a + n/2, b + \sum_{i=1}^n (y_i - \mu)^2/2\right)$

JAGS

Just another Gibbs sampler (JAGS) "is a program for analysis of Bayesian hierarchical models using Markov Chain Monte Carlo (MCMC) simulation not wholly unlike BUGS." We will use JAGS through its R interface `rjags`.

The basic workflow when using `rjags` is

1. Define model and priors in a string
2. Assign data
3. Run JAGS, i.e. simulate from the posterior
4. Summarize as necessary, e.g. mean, median, credible intervals, etc

Binomial model

Let $Y \sim Bin(n, \theta)$ and $\theta \sim Be(1, 1)$ and we observe $y = 3$ successes out of $n = 10$ attempts.

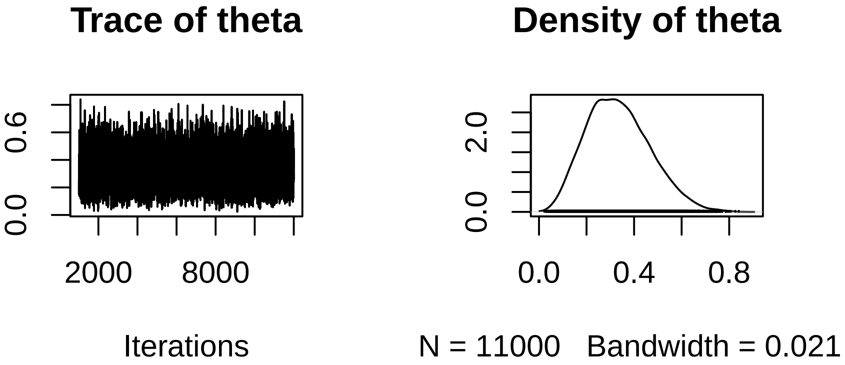
```
model = "
model
{
  y     ~ dbin(theta,n)  # notice p then n
  theta ~ dbeta(a,b)
}
"

dat = list(n=10, y=3, a=1, b=1)

m = jags.model(textConnection(model), dat)

Compiling model graph
   Resolving undeclared variables
   Allocating nodes
Graph information:
   Observed stochastic nodes: 1
   Unobserved stochastic nodes: 1
   Total graph size: 5

Initializing model

r = coda.samples(m, "theta", n.iter = 11000)
```

Binomial model

```
summary(r)


Iterations = 1001:12000
Thinning interval = 1
Number of chains = 1
Sample size per chain = 11000

1. Empirical mean and standard deviation for each variable,
   plus standard error of the mean:

          Mean             SD       Naive SE Time-series SE
      0.334128       0.132603       0.001264       0.001625

2. Quantiles for each variable:

  2.5%    25%    50%    75%  97.5%
0.1075 0.2363 0.3244 0.4216 0.6174
```

Binomial model

```
plot(r)
```



Normal model, unknown mean

Let $Y_i \stackrel{ind}{\sim} N(\mu, s^2)$ and $\mu \sim N(0,1)$.

```
model = "
model
{
  for (i in 1:n) {         # iterate over observations
    y[i] ~ dnorm(mu,1/s2) # precision instead of variance
  }
  mu ~ dnorm(m,1/C)        # cannot use improper prior in JAGS
}
"

dat = list(m=0,C=1,s2=1,y=y)
dat$n = length(dat$y)

m = jags.model(textConnection(model), dat)
```

```
Compiling model graph
   Resolving undeclared variables
   Allocating nodes
Graph information:
   Observed stochastic nodes: 3
   Unobserved stochastic nodes: 1
   Total graph size: 10

Initializing model
```

```
r = coda.samples(m, "mu", n.iter = 11000)
```

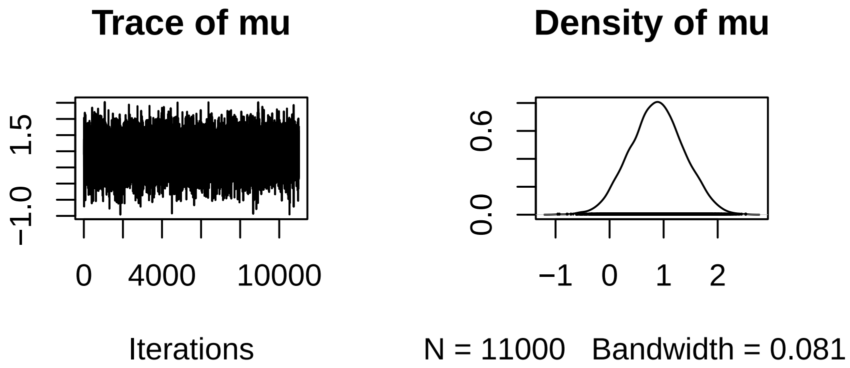
Normal model, unknown mean

```
summary(r)
```

```
Iterations = 1:11000
Thinning interval = 1
Number of chains = 1
Sample size per chain = 11000

1. Empirical mean and standard deviation for each variable,
   plus standard error of the mean:

          Mean             SD       Naive SE Time-series SE
      0.878405       0.497626       0.004745       0.004894

2. Quantiles for each variable:

   2.5%     25%     50%     75%   97.5%
-0.0835  0.5480  0.8779  1.2088  1.8532
```

Normal model, unknown mean

```r
plot(r)
```



Normal model, unknown variance

Let $Y \sim N(m, \sigma^2)$ and $\sigma^2 \sim IG(1, 1)$.

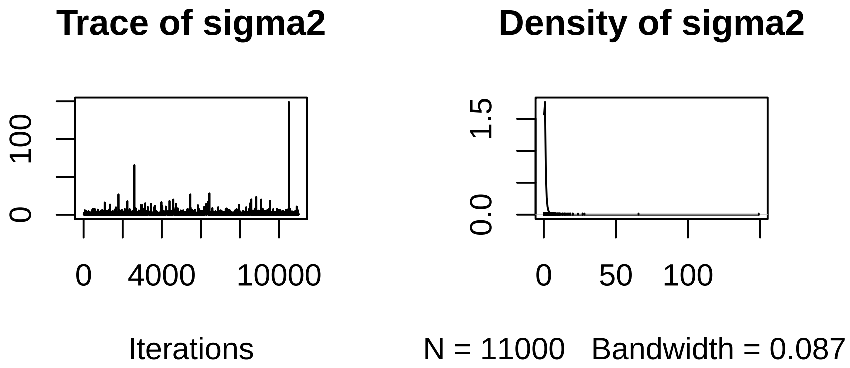
```
model = "
model
{
  for (i in 1:n) {
    y[i] ~ dnorm(m,tau) # precision instead of variance
  }
  tau ~ dgamma(a,b)      # Inverse gamma is not a built in distribution
  sigma2 <- 1/tau        # Functions of parameters
}
"

dat = list(m=1,a=1,b=1,y=y)
dat$n = length(dat$y)

m = jags.model(textConnection(model), dat)


Compiling model graph
   Resolving undeclared variables
   Allocating nodes
Graph information:
   Observed stochastic nodes: 3
   Unobserved stochastic nodes: 1
   Total graph size: 10

Initializing model
```

Normal model, unknown variance

```
summary(r)



Iterations = 1:11000
Thinning interval = 1
Number of chains = 1
Sample size per chain = 11000

1. Empirical mean and standard deviation for each variable,
   plus standard error of the mean:

          Mean             SD       Naive SE Time-series SE
       1.08290        1.99615        0.01903        0.01916

2. Quantiles for each variable:

  2.5%    25%    50%    75%  97.5%
0.2487 0.4901 0.7366 1.2017 3.8753
```

Normal model, unknown variance

```
plot(r)
```



Stan

Stan "is a probabilistic programming language implementing full Bayesian statistical inference." We will use Stan through its R interface `rstan`.

The basic workflow when using `rstan` is (almost exactly the same as for `rjags`):

1. Define model and priors in a string and compile the model.
2. Assign data
3. Run Stan, i.e. simulate from the posterior
4. Summarize as necessary, e.g. mean, median, credible intervals, etc

But, additional coding is required for Stan.

Stan - Binomial model

Let $Y \sim Bin(n, \theta)$ and $\theta \sim Be(1, 1)$.

```
model = "
data {
  int<lower=0> n;              // define range and type
  int<lower=0> a;              // and notice semicolons
  int<lower=0> b;
  int<lower=0,upper=n> y;
}
parameters {
  real<lower=0,upper=1> theta;
}
model {
  y ~ binomial(n,theta);
  theta ~ beta(a,b);
}
"

dat = list(n=10, y=3, a=1, b=1)

m = stan_model(model_code = model) # Only needs to be done once
```

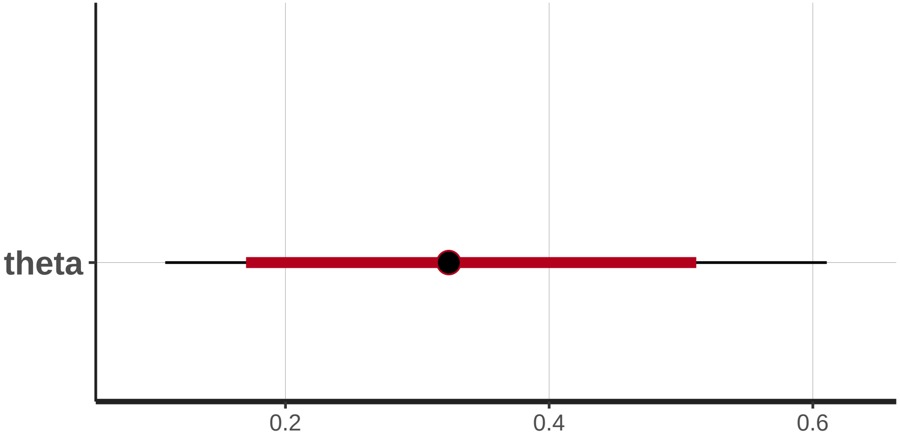
Stan - Binomial model sampling

```r
r = sampling(m, data=dat, iter = 11000, warmup = 1000, refresh=5000)
```

```
r


Inference for Stan model: anon_model.
4 chains, each with iter=11000; warmup=1000; thin=1;
post-warmup draws per chain=10000, total post-warmup draws=40000.

       mean se_mean   sd   2.5%   25%   50%   75% 97.5% n_eff Rhat
theta  0.33    0.00 0.13   0.11  0.24  0.32  0.42  0.61 14380    1
lp__  -8.16    0.01 0.74 -10.29 -8.33 -7.88 -7.69 -7.64 15864    1

Samples were drawn using NUTS(diag_e) at Wed Jan 24 10:50:25 2024.
For each parameter, n_eff is a crude measure of effective sample size,
and Rhat is the potential scale reduction factor on split chains (at
convergence, Rhat=1).
```

Stan - Binomial model

```
plot(r)
```



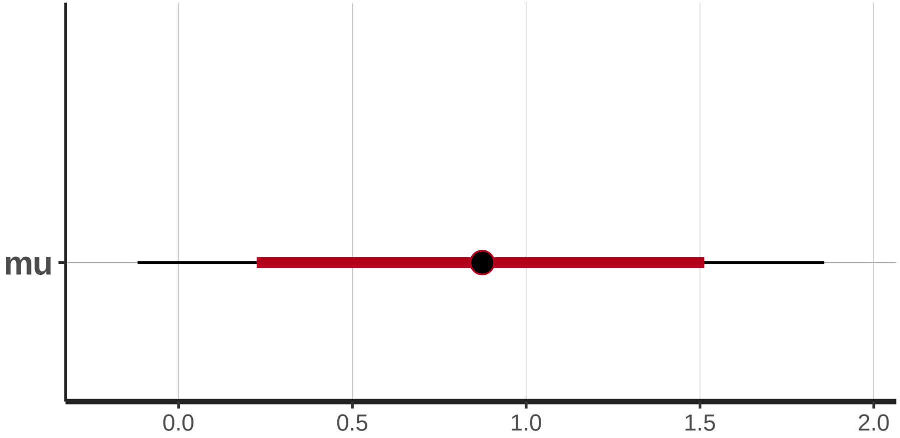
Normal model, unknown mean

Let $Y_i \stackrel{ind}{\sim} N(\mu, s^2)$ and $\mu \sim N(0, 1)$.

```
model = "
data {
  int<lower=0> n;
  real y[n];            // vector
  real<lower=0> s2;
  real m;
  real<lower=0> C;
}
transformed data {      // run once
  real<lower=0> s;
  real<lower=0> sqrtC;
  s     = sqrt(s2);
  sqrtC = sqrt(C);
}
parameters {
  real mu;              // if used alone, implies a uniform prior
}
model {
  y  ~ normal(mu,s);    // vectorized, i.e. assumed independent
  mu ~ normal(m,sqrtC); // standard deviation
}
"

dat = list(m=0,C=1,s2=1,y=y)
dat$n = length(dat$y)
```

Normal model, unknown mean

```
plot(r)
```



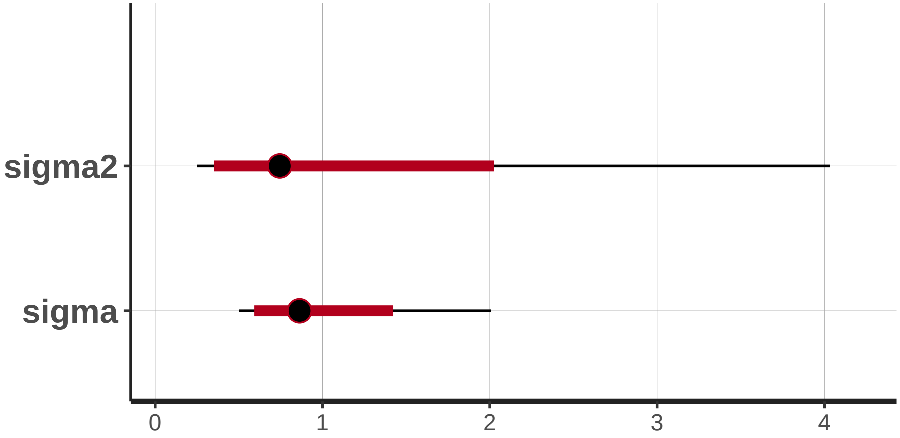
Normal model, unknown variance

Let $Y_i \stackrel{ind}{\sim} N(m,\sigma^2)$ and $\sigma^2 \sim IG(1,1)$.

```
model = "
data {
  int<lower=0> n;
  real y[n];
  real m;
  real<lower=0> a;
  real<lower=0> b;
}
parameters {
  real<lower=0> sigma2;     // if used alone, implies a uniform prior on (0,Inf)
}
transformed parameters {    // deterministic function of parameters
  real<lower=0> sigma;
  sigma = sqrt(sigma2);
}
model {
  y  ~ normal(m,sigma);
  sigma2 ~ inv_gamma(a,b);  // built in inverse gamma distribution
}
"

dat = list(a=1,b=1,m=1,y=y)
dat$n = length(dat$y)

m = stan_model(model_code = model)
```

Normal model, unknown variance

```r
plot(r)
```



Summary

- Normal model
    - Jeffreys prior
    - Conjugate prior
    - Unknown mean
    - Unknown variance
- Bayesian black-box software
    - JAGS, rjags
    - Stan, rstan

Installation

- JAGS: Install both JAGS and rjags
- Stan: Install only rstan OR Install cmdstan and cmdstanr