Name ____

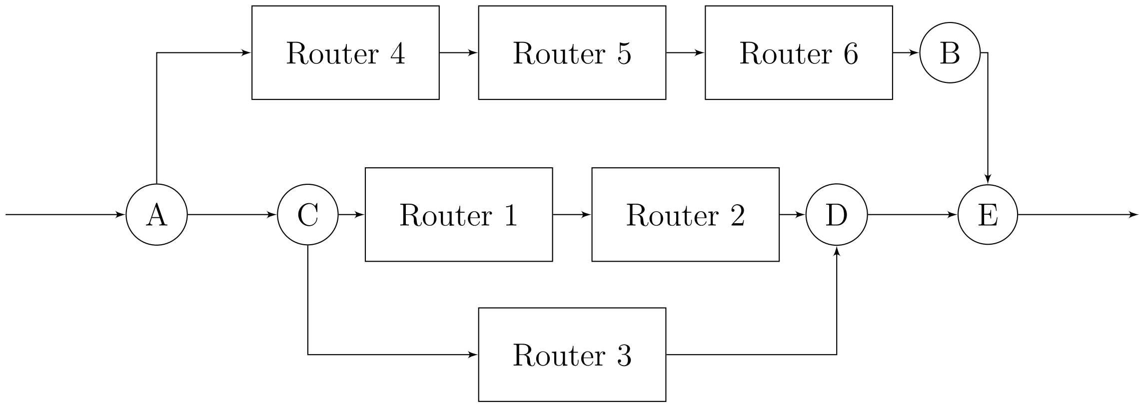
Spring 2017 **STAT 401D** **Exam I**

(100 points)

Instructions:

- Partial credit will be given only if you show your work.
- The questions are not necessarily ordered from easiest to hardest.

1. Consider the system below with all routers acting independently and each having probability 0.80 of successfully transmitting a signal.



   (a) Calculate the probability of a signal passing from A to B. (6 points)

   Answer: The probability of a successful signal from A to B is the product of the probabilities for each component to be successful.

$$0.8^3 \approx 0.512$$

   (b) Calculate the probability of a signal passing from C to D. (6 points)

   Answer: The probability a signal successfully passes through router 1 and 2 is $0.95^2 = 0.64$. Here the signal can travel through router 1 and 2 or through router 3. So the probability of a successful transmission is one minus the probability that both routes are unsuccessful, i.e.

$$1 - (1 - 0.64)(1 - 0.8) \approx 0.928$$

   (c) Calculate the system reliability. (4 points)

   Answer: The reliability is just the probability the system will successfully transmit a signal from A to E which can take the route through 4, 5, and 6 (part a) or a route through 1 and 2 or 3 (part b). Thus

$$1 - (1 - 0.64)(1 - 0.8)(1 - .512) = 1 - (1 - 0.928)(1 - 0.512) \approx 0.96$$

   (d) If you could replace one router with a perfect router that has 100% success which router would it be and why? (4 points)

   Answer: If router 3 is replaced with a perfect router, then all messages will successfully pass through the system.

2. In Colorado, seventy percent of skiers lost during an avalanche are subsequently discovered. Of the skiers that are discovered, 60% have an emergency locator, whereas 90% of the skiers not discovered do not have such a locator. Suppose that a skier has disappeard in an avalanche. If she has an emergency locator, what is the probability that she will be discovered?

   Answer: Let D indicate the event the skier will be discovered and let L indicate the event the skier has an emergency locator. We are given

$$\begin{aligned} P(L|D) &= 0.6 \\ P(L^C|D^C) &= 0.9 \\ P(D) &= 0.70 \end{aligned}$$

   We are asked to find

$$\begin{aligned} P(D|L) &= \frac{P(L|D)P(D)}{P(L|D)P(D)+P(L|D^C)P(D^C)} \\ &= \frac{P(L|D)P(D)}{P(L|D)P(D)+[1-P(L^C|D^C)][1-P(D)]} \\ &= \frac{0.6\times0.7}{0.6\times0.7+[1-0.9][1-0.7]} \\ &= \frac{0.42}{0.42+0.03} = 0.933 \end{aligned}$$

   So the probability the aircraft will be discovered givent that it has an emergency locator, is about 933%.

3. Let X represent a discrete random variable with the following probability mass function

x	1	2	3	4
$P(X = x)$	0.1	0.2	0.3	c

(a) What is the value for c that makes this a valid probability mass function? (6 pts)

Answer: We need

$$\sum_{x=1}^{4} P(X = x) = 1$$

which occurs if $c = P(X = 4) = 0.4$.

(b) What is the expectation of X? (6 pts)

Answer:

$$\begin{aligned} E[X] &= \textstyle\sum_{x=1}^{4} x \cdot P(X = x) \\ &= 1 \times 0.1 + 2 \times 0.2 + 3 \times 0.3 + 4 \times 0.4 \\ &= 0.1 + 0.4 + 0.9 + 1.6 \\ &= 3 \end{aligned}$$

(c) What is the variance of X? (7 pts)

Answer:

$$\begin{aligned} Var[X] &= \textstyle\sum_{x=1}^{4} (x - \mu)^2 P(X = x) \\ &= (1-3)^2 \times 0.1 + (2-3)^2 \times 0.2 + (3-3)^2 \times 0.3 + (4-3)^2 \times 0.4 \\ &= 4 \times 0.1 + 1 \times 0.2 + 0 \times 0.3 + 1 \times 0.4 \\ &= 0.4 + 0.2 + 0 + 0.4 = 1 \end{aligned}$$

4. Suppose we are performing an experiment to study the effect of skin lotions. Participants have dry skin and Lotion A is randomly assigned to one arm and Lotion B is then used on the other arm. For each participant, we record whether Lotion A worked better than Lotion B. Since have no reason to believe which lotion is better, we assume the probability that A works better than B is 0.5. For the following questions, assume there are 10 participants and the response from each participant is independent.

   (a) What is the probability that exactly 2 of the 10 responses indicate that Lotion A worked better than Lotion B? (6 points)

   Answer: Let Y be the number of participants that indicate Lotion A worked better than Lotion B and assume $Y \sim Bin(n,p)$ with $n = 10$ and $p = 0.5$.
   Thus we have

$$P(Y = 2) = \binom{10}{2} 0.5^2 (1 - 0.5)^{10-2} = 0.0439.$$

```r
n <- 10
p <- 0.5
dbinom(2, size = n, prob = p)

## [1] 0.04394531
```

   (b) What is the probability that 2 or fewer of the response indicate that Lotion A worked better than Lotion B? (7 points)

   Answer: We have

$$P(Y \leq 2) = \sum_{y=0}^{2} \binom{10}{y} 0.5^y (1 - 0.5)^{10-y} = 0.0547.$$

```r
pbinom(2, size = n, prob = p)

## [1] 0.0546875

sum(dbinom(0:2, size = n, prob = p))

## [1] 0.0546875
```

   (c) What is the probability between 3 and 7 responses (inclusive, i.e. 3 and 7 are included) indicate that Lotion A worked better than Lotion B? (7 points)

   Answer: We have

$$P(3 \leq Y \leq 7) = \sum_{y=3}^{7} \binom{10}{y} 0.5^y (1 - 0.5)^{10-y} = 0.0547.$$

   or, equivalently

$$P(3 \leq Y \leq 7) = P(Y \leq 7) - P(Y < 3) = P(Y \leq 7) - P(Y \leq 2)$$

```r
sum(dbinom(3:7, size = n, prob = p))

## [1] 0.890625

pbinom(7, size = n, prob = p) - pbinom(2, size = n, prob = p)

## [1] 0.890625
```

5. A network of 100 temperature sensors is deployed to measure the ocean temperature near an underwater volcano. The sensors act independently and have an expected value that is actual temperature with a variance of 0.36^oC. The sensors each report to a single hub and the hub reports to a base station if at least one of the sensors reports a temperature more than 2^oC different from the last reported sensor.

   (a) Suppose the last reported temperature was 80.1^oC and the current actual temperature is 80.3^oC. What is the probability the hub will report to the base station? (10 points)

   Answer: Let X_i be the temperature reported by sensor i for $i = 1, \ldots, n$. Assume $E[X_i] = 80.3^oC$ and $Var[X_i] = 0.4^oC$.

$$\begin{aligned} P(\text{hub reports}) &= P(\text{at least one } X_i < 78.1 \text{ or } X_i > 82.1) \\ &= 1 - P(\text{all } 78.1 < X_i < 82.1) \\ &= 1 - P(78.1 < X_i < 82.1)^n && \text{independence} \\ &= 1 - [P(X_i < 82.1) - P(X_i < 78.1)]^n \end{aligned}$$

```r
n <- 100
mu <- 80.3
sigma <- sqrt(0.36)
1-(pnorm(82.1, mean = mu, sd = sigma)-pnorm(78.1, mean = mu, sd = sigma))^n

## [1] 0.1370383
```

   (b) Since communication to the base station is the battery draining operation, the scientist set the hub to only report to the base station if the average of all sensors is more than 0.4^oC different from the previous report. Using the temperatures in the previous problem, what is the probability the hub will report to the base station? (10 points)

   Answer: Using the notation in the previous answer, we have $\overline{X} \sim N(80.3, 0.6/10)$.

$$\begin{aligned} P(\text{hub reports}) &= P(\overline{X} < 79.7 \text{ or } \overline{X} > 80.5) \\ &= 1 - P(79.7 < \overline{X} < 80.5) \\ &= 1 - [P(X_i < 80.5) - P(X_i < 79.7)] \end{aligned}$$

```r
1-(pnorm(80.5, mean = mu, sd = sigma/10)-pnorm(79.7, mean = mu, sd = sigma/10))

## [1] 0.0004290603
```