I4 - Bayesian parameter estimation in a normal model

STAT 5870 (Engineering)
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Bayesian parameter estimation

Recall that Bayesian parameter estimation involves

$$p(\theta|y) = \frac{p(y|\theta)p(\theta)}{p(y)} = \frac{p(y|\theta)p(\theta)}{\int p(y|\theta)p(\theta)d\theta}$$

with

- posterior $p(\theta|y)$,
- prior $p(\theta)$,
- model $p(y|\theta)$, and
- prior predictive $p(y)$.

For this video, $\theta = (\mu, \sigma^2)$ and

$$y|\mu, \sigma^2 \sim N(\mu, \sigma^2).$$

Bayesian parameter estimation in a normal model

Let $Y_i \stackrel{ind}{\sim} N(\mu, \sigma^2)$ and the default prior

$$p(\mu, \sigma^2) \propto \frac{1}{\sigma^2}.$$

Note: This "prior" is not a distribution since its integral is not finite. Nonetheless, we can still derive the following posterior

$$\mu|y \sim t_{n-1}(\overline{y}, s^2/n) \qquad \text{and} \qquad \sigma^2|y \sim IG\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$$

where

- n is the sample size,
- $\overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$ is the sample mean, and
- $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \overline{y})^2$ is the sample variance.

Posterior for the mean

The posterior for the mean is

$$\mu|y \sim t_{n-1}(\overline{y}, s^2/n)$$

and from properties of the generalized Student's t distribution, we know

- $E[\mu|y] = \overline{y}$ for $n > 2$,
- $Var[\mu|y] = \frac{(n-1)s^2}{(n-3)} \Big/ n$ for $n > 3$,

and

$$\frac{\mu - \overline{y}}{s/\sqrt{n}} \sim t_{n-1}.$$

Credible intervals for μ

Since

$$\frac{\mu - \overline{y}}{s/\sqrt{n}} \sim t_{n-1}$$

a $100(1-a)\%$ equal-tail credible interval is

$$\overline{y} \pm t_{n-1,a/2}\, s/\sqrt{n}$$

where $t_{n-1,a/2}$ is a t critical value such that $P(T_{n-1} < t_{n-1,a/2}) = 1 - a/2$ when $T_{n-1} \sim t_{n-1}$.

For example, $t_{10-1,0.05/2}$ is

```
n = 10
a = 0.05 # 95\% CI
qt(1-a/2, df = n-1)

[1] 2.262157
```

Posterior for the variance

The posterior for the mean is

$$\sigma^2|y \sim IG\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$$

and from properties of the inverse Gamma distribution, we know

- $E[\sigma^2|y] = \frac{(n-1)s^2}{n-3}$ for $n > 3$,

and

$$\frac{1}{\sigma^2}\bigg| y \sim Ga\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$$

where $(n-1)s^2/2$ is the rate parameter.

Credible intervals for σ^2

For a $100(1-a)\%$ credible interval, we need

$$a/2 = P(\sigma^2 < L|y) = P(\sigma^2 > U|y).$$

To do this, we will find

$$a/2 = P\left(\frac{1}{\sigma^2} > \frac{1}{L}\middle| y\right) = P\left(\frac{1}{\sigma^2} < \frac{1}{U}\middle| y\right).$$

Here is a function that performs this computation

```
qinvgamma <- function(p, shape, scale = 1)
  1/qgamma(1-p, shape = shape, rate = scale)
```

Posterior for the standard deviation, σ

The variance is hard to interpret because its units are squared relative to Y_i. In contrast, the standard deviation $\sigma = \sqrt{\sigma^2}$ units are the same as Y_i.

For credible intervals (or any quantile), we can compute the square root of the endpoints since

$$P(\sigma^2 < c^2) = P(\sigma < c).$$

Find the pdf through transformations of random variables. In R code,

```
dinvgamma <- function(x, shape, scale = 1)
  dgamma(1/x, shape = shape, rate = scale)/x^2

dsqrtinvgamma = function(x, shape, scale)
  dinvgamma(x^2, shape, scale)*2*x
```

Yield data

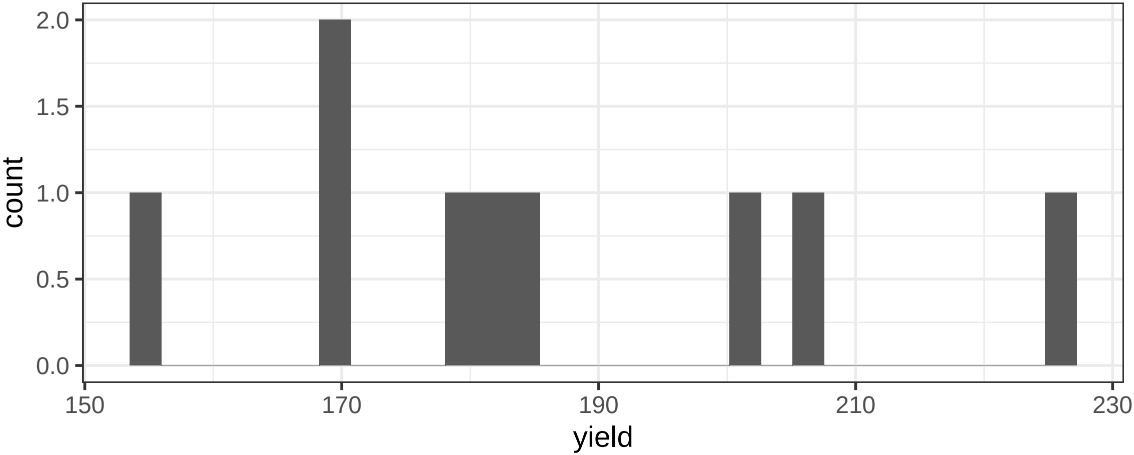
Suppose we have a random sample of 9 Iowa farms and we obtain corn yield in bushels per acre on those farms. Let Y_i be the yield for farm i in bushels/acre and assume

$$Y_i \stackrel{ind}{\sim} N(\mu, \sigma^2).$$

We are interested in making statements about μ and σ^2.

```
yield_data <- read.csv("yield.csv")
nrow(yield_data)

[1] 9

yield_data

   farm    yield
1 farm1 153.5451
2 farm2 205.6999
3 farm3 178.7548
4 farm4 170.1692
5 farm5 224.7723
6 farm6 184.0806
7 farm7 169.8615
8 farm8 201.2721
9 farm9 181.6356
```

Histogram of yield



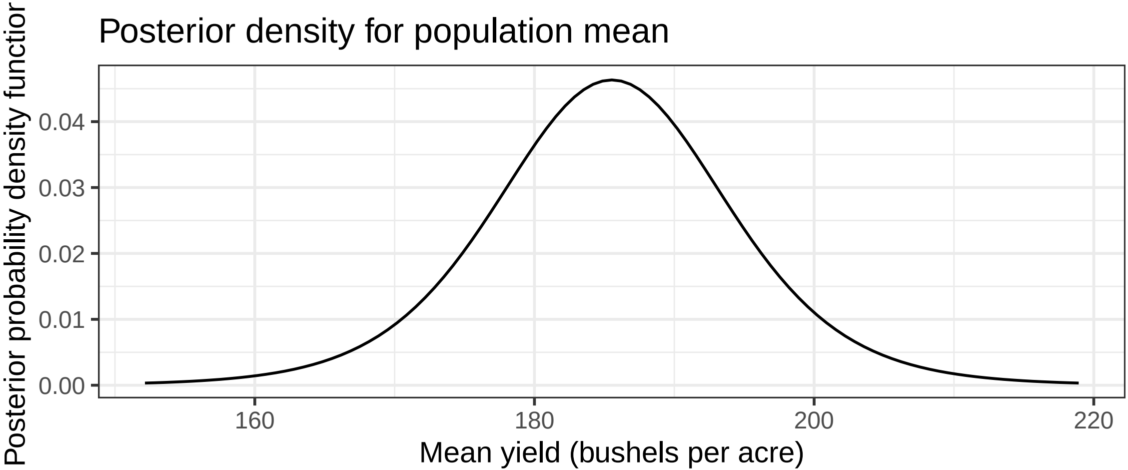
Calculate sufficient statistics

```
(n               <- length(yield_data$yield))

[1] 9

(sample_mean     <- mean(yield_data$yield))

[1] 185.5323

(sample_variance <- var(yield_data$yield))

[1] 470.2817
```

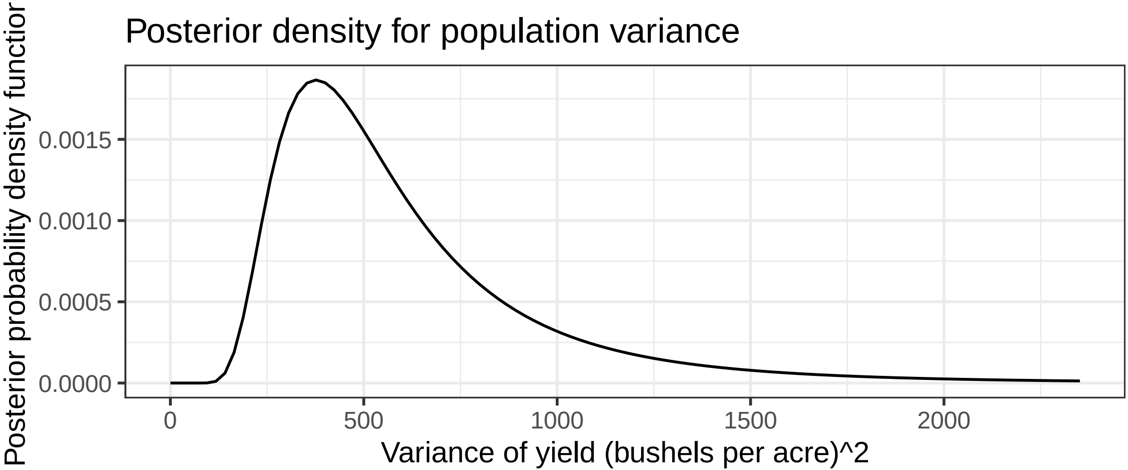
Use these sufficient statistics to calculate:

- posterior densities
- posterior means
- credible intervals

Posterior density for μ



Posterior density for σ^2



Posterior means

```r
# Posterior mean of population yield mean, E[mu|y]
sample_mean
```

```
[1] 185.5323
```

Posterior mean for μ is $E[\mu|y] = 186$ bushels/acre.

```r
# Posterior mean of population yield variance
post_mean_var = (n-1)*sample_variance / (n-3)
post_mean_var
```

```
[1] 627.0422
```

Posterior mean for σ^2 is $E[\sigma^2|y] = 627$ (bushels/acre)^2.

Credible intervals

```
# 95% credible interval for the population mean
a = 0.05
mean_ci = sample_mean + c(-1,1) *  qt(1-a/2, df = n-1) * sqrt(sample_variance/n)
mean_ci
```

```
[1] 168.8630 202.2017
```

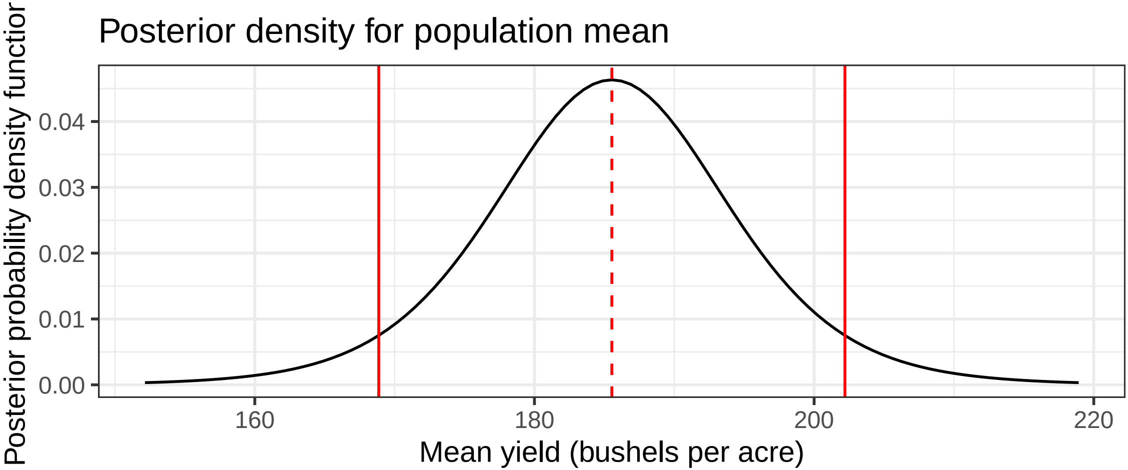
So a 95% credible interval for μ is (169,202) bushels/acre.

```
# 95% credible interval for the population variance
var_ci = qinvgamma(c(a/2, 1-a/2),
                   shape = (n-1)/2,
                   scale = (n-1)*sample_variance/2)
var_ci
```

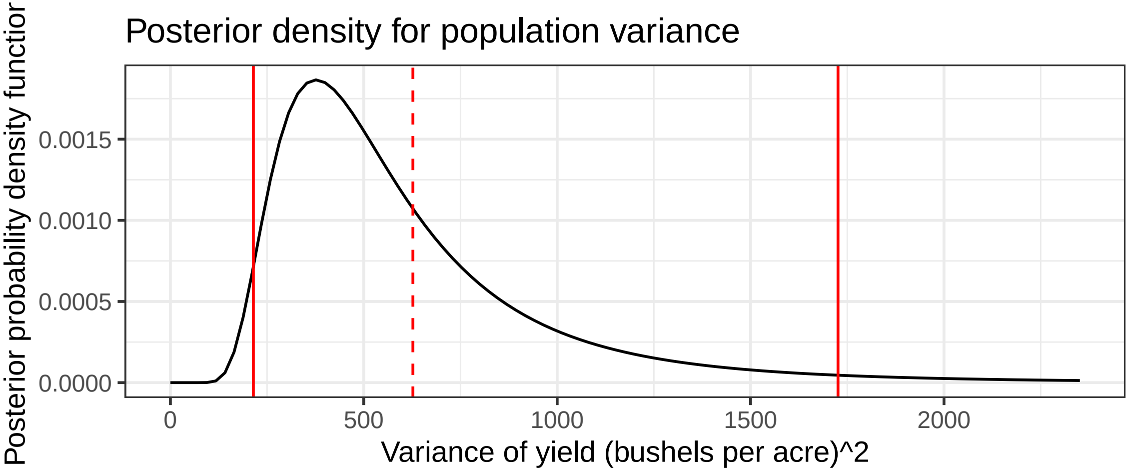
```
[1]  214.5623 1726.0175
```

So a 95% credible interval for σ^2 is (215,1726) (bushels/acre)^2.

Posterior density for μ



Posterior density for σ^2



Posterior for the standard deviation, σ

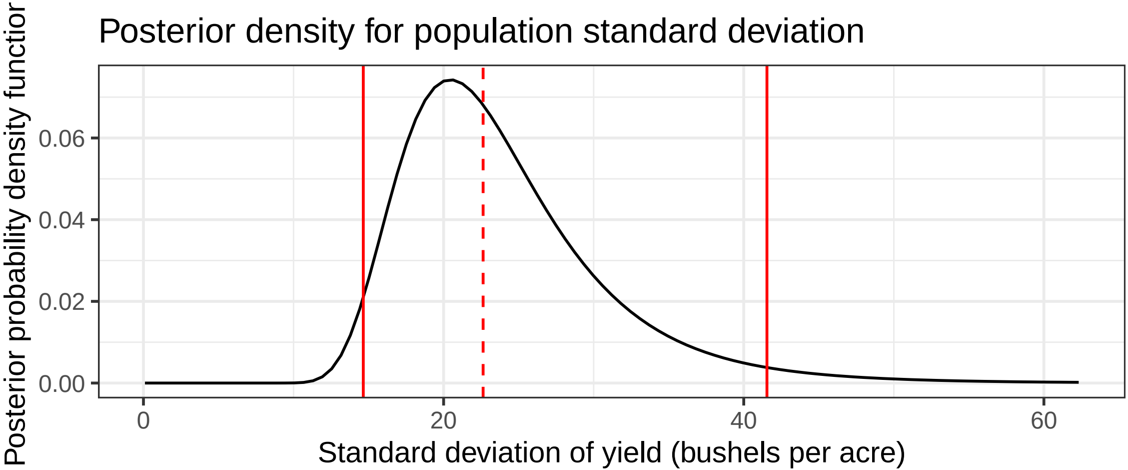
```
# Posterior median and 95% CI for population yield standard deviation
sd_median = sqrt(qinvgamma(.5, shape = (n-1)/2, scale = (n-1)*sample_variance/2))
sd_median

[1] 22.63362
```

So the posterior median for σ is 23 bushels/acre.

```
# Posterior 95% CI for the population yield standard deviation
sd_ci = sqrt(var_ci)
sd_ci

[1] 14.64795 41.54537
```

So a posterior 95% credible interval for σ is (15, 42) bushels/acre.

Posterior for the standard deviation, σ



Bayesian inference in a normal model

- Prior: $p(\mu, \sigma^2) = 1/\sigma^2$
- Posterior:

$$\mu|y \sim t_{n-1}(\overline{y}, s^2/n) \qquad \text{and} \qquad \sigma^2|y \sim IG\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$$

```
# Sufficient statistics
n               <- length(y)
sample_mean     <- mean(y)
sample_variance <- var(y)

# Posterior expectations
sample_mean                        # mu
(n-1)*sample_variance / (n-3) # sigma^2

# Posterior medians
var_median <- qinvgamma(.5, shape = (n-1)/2, scale = (n-1)*sample_variance/2)
sd_median  <- sqrt(var_median)

# Posterior credible intervals
sample_mean + c(-1,1) *  qt(1-a/2, df = n-1) * sqrt(sample_variance/n)
var_ci <- qinvgamma(c(a/2,1-a/2), shape = (n-1)/2, scale = (n-1)*sample_variance/2)
sd_ci  <- sqrt(var_ci)
```