# I05 - Confidence intervals

STAT 5870 (Engineering)
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# Exact confidence intervals

The coverage of an interval estimator is the probability the interval will contain the true value of the parameter *when the data are considered to be random*. If an interval estimator has $100(1-a)\%$ coverage, then we call it a $100(1-a)\%$ confidence interval and $1-a$ is the confidence level.

That is, we calculate

$$1 - a = P(L < \theta < U)$$

where $L$ and $U$ are random because they depend on the data. Thus confidence is a statement about the procedure.

# Normal model

If $Y_i \stackrel{ind}{\sim} N(\mu, \sigma^2)$ and we assume the default prior $p(\mu, \sigma^2) \propto 1/\sigma^2$, then a $100(1-a)\%$ credible interval for $\mu$ is given by

$$\overline{y} \pm t_{n-1,a/2} s/\sqrt{n}.$$

When the data are considered random

$$T_{n-1} = \frac{\overline{Y} - \mu}{S/\sqrt{n}} \sim t_{n-1}(0,1)$$

thus the probability $\mu$ is within our credible interval is

$$\begin{array}{rl} & P\left(\overline{Y} - t_{n-1,a/2} S/\sqrt{n} < \mu < \overline{Y} + t_{n-1,a/2} S/\sqrt{n}\right) \\ & = P\left(-t_{n-1,a/2} < \frac{\overline{Y}-\mu}{S/\sqrt{n}} < t_{n-1,a/2}\right) \\ & = P\left(-t_{n-1,a/2} < T_{n-1} < t_{n-1,a/2}\right) \\ & = 1 - a. \end{array}$$

Thus, this $100(1-a)\%$ credible interval is also a $100(1-a)\%$ confidence interval.

# Yield data example

Recall the corn yield example from I04 with $9$ randomly selected fields in Iowa whose sample average yield is $186$ and sample standard deviation is $22$. Then a $95\%$ confidence interval for the mean corn yield on Iowa farms is

$$186 \pm 2.31 \times 22/\sqrt{9} = (169, 202).$$

# Standard error

The standard error of an estimator is an *estimate* of the standard deviation of the estimator (when the data are considered random).

If $Y \sim Bin(n, \theta)$, then

$$\hat\theta = \frac{Y}{n} \qquad \text{has} \qquad SE[\hat\theta] = \sqrt{\frac{\hat\theta(1-\hat\theta)}{n}}.$$

If $Y_i \stackrel{ind}{\sim} N(\mu, \sigma^2)$, then

$$\hat\mu = \overline{Y} \qquad \text{has} \qquad SE[\hat\mu] = S/\sqrt{n}.$$

# Approximate confidence intervals

If an unbiased estimator has an asymptotic normal distribution, then we can construct an approximate $100(1-a)\%$ confidence interval for $E[\hat\theta] = \theta$ using

$$\hat\theta \pm z_{a/2} SE[\hat\theta].$$

where $SE[\hat\theta]$ is the standard error of the estimator and $P(Z > z_{a/2}) = a/2$.

This comes from the fact that if $\hat\theta \overset{.}{\sim} N(\theta, SE[\hat\theta]^2)$, then

$$\begin{aligned}
&P\left(\hat\theta - z_{a/2}SE(\hat\theta) < \theta < \hat\theta + z_{a/2}SE(\hat\theta)\right) \\
&= P\left(-z_{a/2} < \frac{\hat\theta-\theta}{SE(\hat\theta)} < z_{a/2}\right) \\
&\approx P\left(-z_{a/2} < Z < z_{a/2}\right) \\
&= 1-a.
\end{aligned}$$

# Normal example

If $Y_i \stackrel{ind}{\sim} N(\mu, \sigma^2)$ and we have the estimator $\hat{\mu} = \overline{Y}$, then

$$E[\hat{\mu}] = \mu \qquad \text{and} \qquad SE[\hat{\mu}] = S/\sqrt{n}$$

Thus an approximate $100(1-a)\%$ confidence interval for $\mu = E[\hat{\mu}]$ is

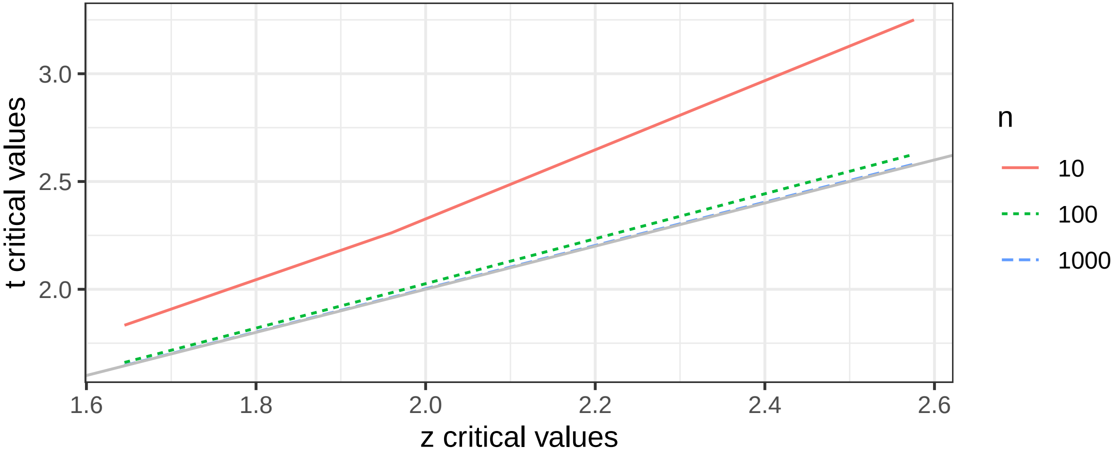
$$\hat{\mu} \pm z_{a/2} SE[\hat{\mu}] = \overline{Y} \pm z_{a/2} S/\sqrt{n}.$$

Note that this is almost identical to the exact $100(1-a)\%$ confidence interval for $\mu$,

$$\overline{Y} \pm t_{n-1,a/2} S/\sqrt{n}$$

and when $n$ is large $z_{a/2} \approx t_{n-1,a/2}$.

# T critical values vs Z critical values



# Approximate confidence interval for binomial proportion

If $Y \sim Bin(n, \theta)$, then an approximate $100(1-a)\%$ confidence interval for $\theta$ is

$$\hat{\theta} \pm z_{a/2}\sqrt{\frac{\hat{\theta}(1-\hat{\theta})}{n}}.$$

where $\hat{\theta} = Y/n$ since

$$E[\hat{\theta}] = E\left[\frac{Y}{n}\right] = \theta$$

and

$$SE[\hat{\theta}] = \sqrt{\frac{\hat{\theta}(1-\hat{\theta})}{n}}.$$

# Gallup poll example

In a Gallup poll dated 2017/02/19, 32.1% of respondents of the 1,500 randomly selected U.S. adults indicated that they were "engaged at work". Thus an approximate 95% confidence interval for the proportion of all U.S. adults is

$$0.321 \pm 1.96 \times \sqrt{\frac{.321(1-.321)}{1500}} = (0.30, 0.34).$$

# Confidence interval summary

| Model | Parameter | Estimator | Confidence Interval | Type |
|---|---|---|---|---|
| $Y_i \stackrel{ind}{\sim} N(\mu, \sigma^2)$ | $\mu$ | $\hat{\mu} = \overline{y}$ | $\hat{\mu} \pm t_{n-1,a/2} s/\sqrt{n}$ | exact |
| $Y_i \stackrel{ind}{\sim} N(\mu, \sigma^2)$ | $\mu$ | $\hat{\mu} = \overline{y}$ | $\hat{\mu} \pm z_{a/2} s/\sqrt{n}$ | approximate |
| $Y \sim Bin(n, \theta)$ | $\theta$ | $\hat{\theta} = y/n$ | $\hat{\theta} \pm z_{a/2}\sqrt{\hat{\theta}(1-\hat{\theta})/n}$ | approximate |
| $Y_i \stackrel{ind}{\sim} Ber(\theta)$ | $\theta$ | $\hat{\theta} = \overline{y}$ | $\hat{\theta} \pm z_{a/2}\sqrt{\hat{\theta}(1-\hat{\theta})/n}$ | approximate |

The Bayesian credible intervals we discuss provide approximate confidence intervals. For example, for binomial data the following is an approximate confidence interval:

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qbeta(c(a/2, 1-a/2), 1 + y, 1 + n - y)
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Approximate means that the coverage will get closer to the desired probability, i.e. $100(1-a)\%$, as the sample size gets larger.