R01 - Simple linear regression

STAT 5870 (Engineering)
Iowa State University

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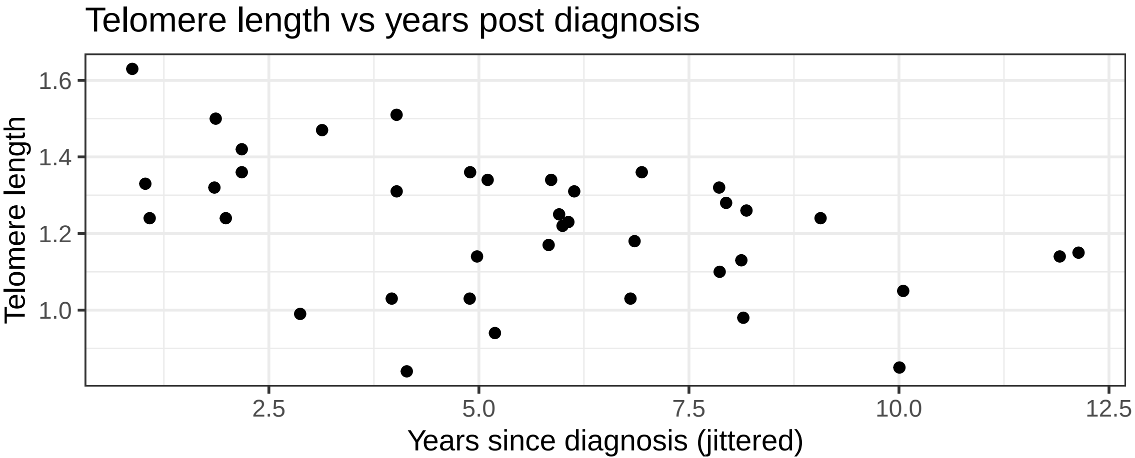
Telomere length

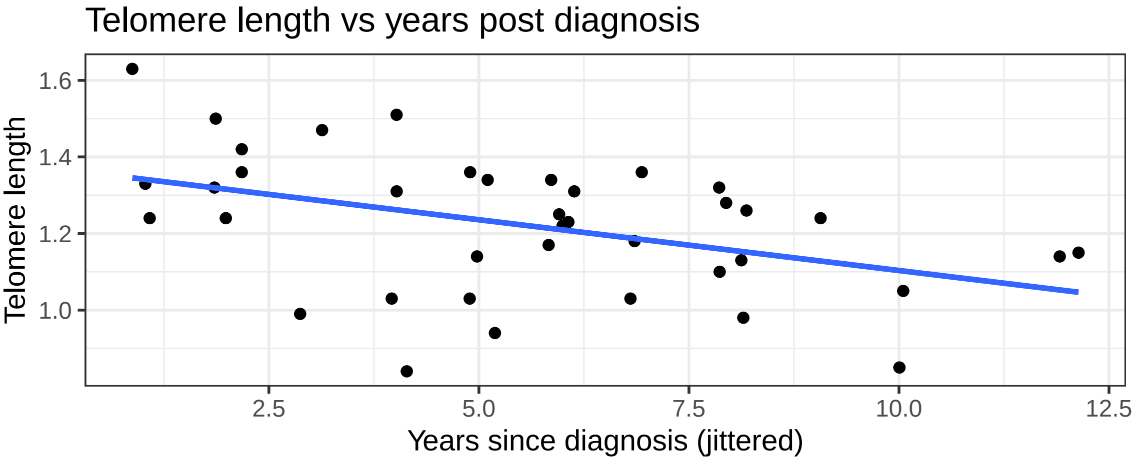
http://www.pnas.org/content/101/49/17312

> *People who are stressed over long periods tend to look haggard, and it is commonly thought that psychological stress leads to premature aging [as measured by decreased telomere length]*
>
> *...*
>
> *examine the importance of ... caregiving stress (...number of years since a child's diagnosis [of a chronic disease]) [on telomere length]*
>
> *...*
>
> *Telomere length values were measured from DNA by a quantitative PCR assay that determines the relative ratio of telomere repeat copy number to single-copy gene copy number (T/S ratio) in experimental samples as compared with a reference DNA sample.*

Data


Telomere length vs years post diagnosis
Telomere length
1.6
1.4
1.2
1.0
2.5
5.0
7.5
10.0
12.5
Years since diagnosis (jittered)

Data with regression line



Simple Linear Regression

The simple linear regression model is

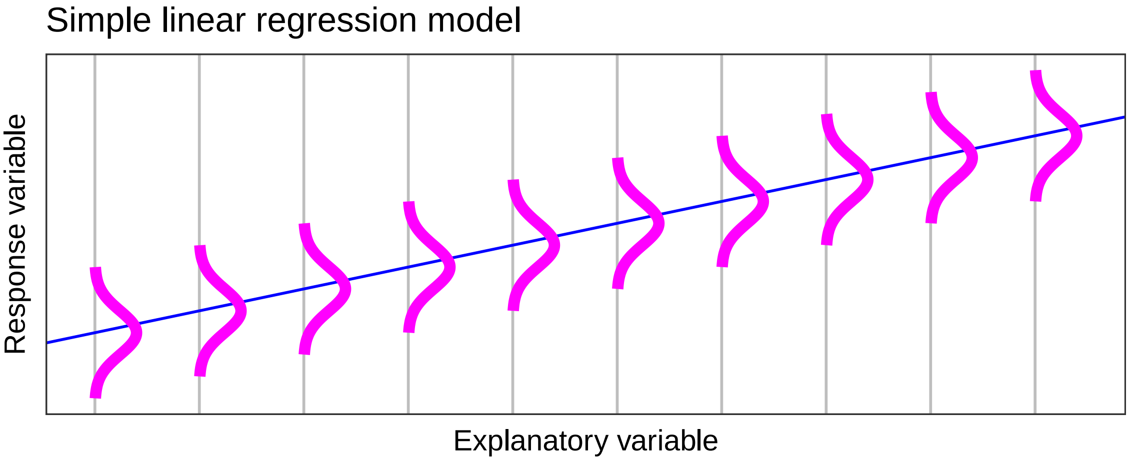
$$Y_i \stackrel{ind}{\sim} N(\beta_0 + \beta_1 X_i, \sigma^2)$$

where Y_i and X_i are the response and explanatory variable, respectively, for individual i.

Terminology (all of these are equivalent):

response	explanatory
outcome	covariate
dependent	independent
endogenous	exogenous

Simple linear regression - visualized



Parameter interpretation

Recall:

$$E[Y_i|X_i = x] = \beta_0 + \beta_1 x \qquad Var[Y_i|X_i = x] = \sigma^2$$

- If $X_i = 0$, then $E[Y_i|X_i = 0] = \beta_0$.

  β_0 is the expected response when the explanatory variable is zero.

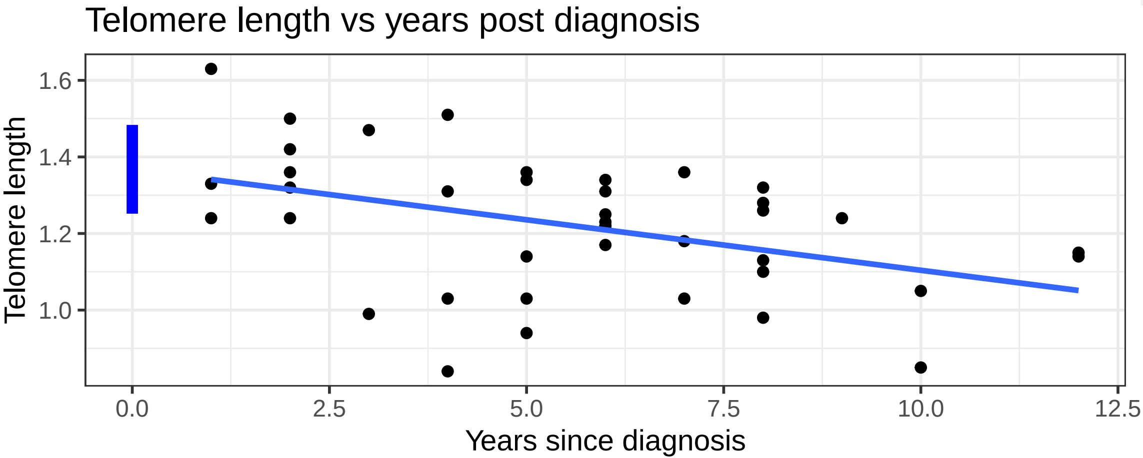
- If X_i increases from x to $x + 1$, then

$$\begin{array}{rl} E[Y_i|X_i = x + 1] & = \beta_0 + \beta_1 x + \beta_1 \\ - E[Y_i|X_i = x \quad\;\;] & = \beta_0 + \beta_1 x \\ \hline & = \qquad\qquad\quad \beta_1 \end{array}$$

  β_1 is the expected increase in the response for each unit increase in the explanatory variable.

- σ is the standard deviation of the response for a fixed value of the explanatory variable.

Simple linear regression - visualized



Errors v residuals

Remove the mean:

$$Y_i = \beta_0 + \beta_1 X_i + e_i \qquad e_i \stackrel{iid}{\sim} N(0, \sigma^2)$$

So the error is

$$e_i = Y_i - (\beta_0 + \beta_1 X_i)$$

which we approximate by the residual

$$r_i = \hat{e}_i = Y_i - (\hat{\beta}_0 + \hat{\beta}_1 X_i)$$

These residuals we will use for a number of purposes including

- assessing model assumptions,
- identifying outliers, and
- estimating error variance.

Estimators

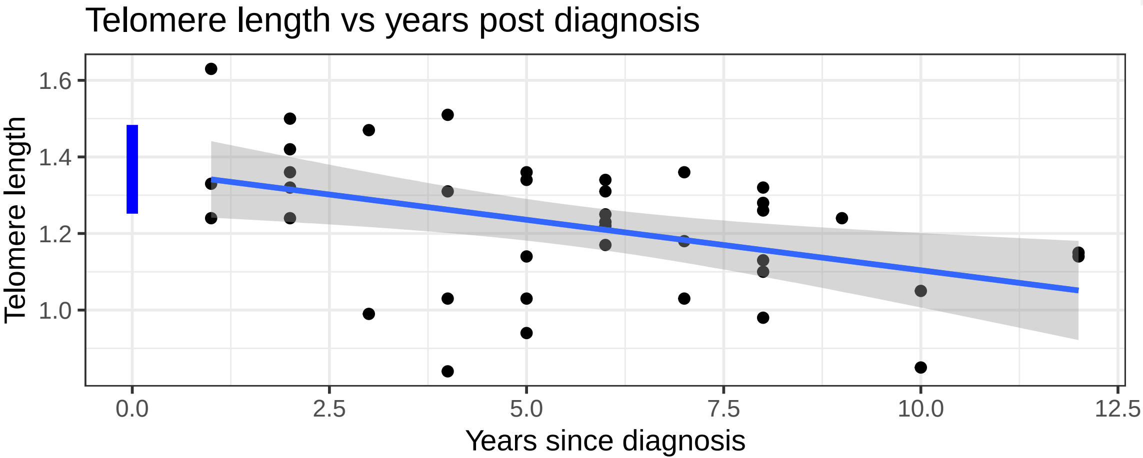
The least squares (minimize $\sum_{i=1}^{n} r_i^2$), maximum likelihood, and Bayesian estimators (prior $1/\sigma^2$) are

$$\begin{aligned}
\hat{\beta}_1 &= SXY/SXX \\
\hat{\beta}_0 &= \overline{Y} - \hat{\beta}_1 \overline{X} \\
\hat{\sigma}^2 &= SSE/(n-2) \qquad df = n-2
\end{aligned}$$

$$\begin{aligned}
\overline{X} &= \frac{1}{n}\sum_{i=1}^{n} X_i \\
\overline{Y} &= \frac{1}{n}\sum_{i=1}^{n} Y_i
\end{aligned}$$

$$\begin{aligned}
SXY &= \sum_{i=1}^{n}(X_i - \overline{X})(Y_i - \overline{Y}) \\
SXX &= \sum_{i=1}^{n}(X_i - \overline{X})(X_i - \overline{X}) = \sum_{i=1}^{n}(X_i - \overline{X})^2 \\
SSE &= \sum_{i=1}^{n} r_i^2
\end{aligned}$$

Residuals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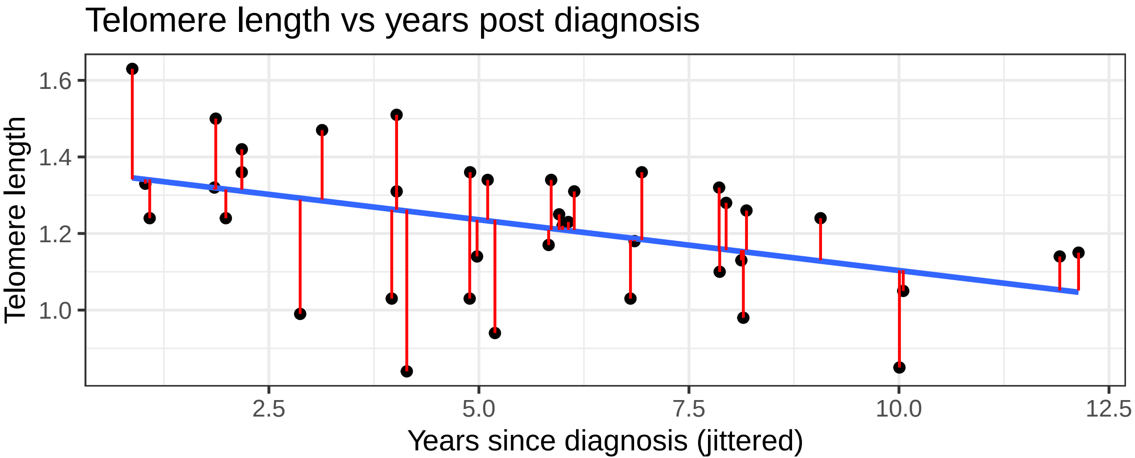
Years since diagnosis

Residuals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Years since diagnosis (jittered)

How certain are we about $\hat{\beta}_0$ and $\hat{\beta}_1$?

We quantify this uncertainty using their standard errors (or posterior scale parameters):

$$
\begin{aligned}
SE(\hat{\beta}_0) &= \hat{\sigma}\sqrt{\frac{1}{n} + \frac{\overline{X}^2}{(n-1)s_X^2}} && df = n-2 \\
SE(\hat{\beta}_1) &= \hat{\sigma}\sqrt{\frac{1}{(n-1)s_X^2}} && df = n-2
\end{aligned}
$$

$$
\begin{aligned}
s_X^2 &= SXX/(n-1) \\
s_Y^2 &= SYY/(n-1) \\
SYY &= \sum_{i=1}^{n}(Y_i - \overline{Y})^2
\end{aligned}
$$

$$
\begin{aligned}
r_{XY} &= \frac{SXY/(n-1)}{s_X s_Y} && \text{correlation coefficient} \\
R^2 &= r_{XY}^2 = \frac{SST-SSE}{SST} && \text{coefficient of determination} \\
SST &= SYY = \sum_{i=1}^{n}(Y_i - \overline{Y})^2
\end{aligned}
$$

The coefficient of determination (R^2) is the proportion of the total response variation explained by the model.

Default Bayesian analysis of the simple linear regression model

If we assume the default prior $p(\beta_0, \beta_1, \sigma^2) \propto 1/\sigma^2$, then the marginal posteriors for the mean parameters are

$$\beta_j | y \sim t_{n-2}(\hat{\beta}_j, SE(\hat{\beta}_j)^2).$$

We can construct a $100(1-a)\%$ two-sided credible interval for β_j via

$$\hat{\beta}_j \pm t_{n-2,1-a/2} SE(\hat{\beta}_j)$$

where $P(T_{n-2} < t_{n-2,1-a/2}) = 1 - a/2$ for $T_{n-2} \sim t_{n-2}$.

We can compute posterior probabilities via

$$P(\beta_j > b_j | y) = P\left(T_{n-2} > \frac{b_j - \hat{\beta}_j}{SE(\hat{\beta}_j)}\right) \quad \text{or} \quad P(\beta_j < b_j | y) = P\left(T_{n-2} < \frac{b_j - \hat{\beta}_j}{SE(\hat{\beta}_j)}\right)$$

often $b_j = 0$.

p-values and confidence interval

We can construct a $100(1-a)\%$ two-sided confidence interval for β_j via

$$\hat{\beta}_j \pm t_{n-2,1-a/2} SE(\hat{\beta}_j).$$

We can compute one-sided p-values,

$H_0 : \beta_j \geq b_j$ vs $H_A : \beta_j < b_j$ has

$$p\text{-value} = P\left(T_{n-2} < \frac{\hat{\beta}_j - b_j}{SE(\hat{\beta}_j)}\right)$$

and $H_0 : \beta_j \leq b_j$ vs $H_A : \beta_j > b_j$ has

$$p\text{-value} = P\left(T_{n-2} > \frac{\hat{\beta}_j - b_j}{SE(\hat{\beta}_j)}\right)$$

or two-sided p-values for $H_0 : \beta_j = b_j$ vs $H_A : \beta_j \neq b_j$:

$$= 2 \times \min\left\{P\left(T_{n-2} > \frac{\hat{\beta}_j - b_j}{SE(\hat{\beta}_j)}\right), P\left(T_{n-2} < \frac{\hat{\beta}_j - b_j}{SE(\hat{\beta}_j)}\right)\right\} = 2 \times P\left(T_{n-2} < -\left|\frac{\hat{\beta}_j - b_j}{SE(\hat{\beta}_j)}\right|\right)$$

software default is usually $b_j = 0$.

Calculations “by hand” in R

```
n    = nrow(Telomeres)
Xbar = mean(Telomeres$years)
Ybar = mean(Telomeres$telomere.length)
s_X  = sd(Telomeres$years)
s_Y  = sd(Telomeres$telomere.length)
r_XY = cor(Telomeres$telomere.length, Telomeres$years)

SXX = (n-1)*s_X^2
SYY = (n-1)*s_Y^2
SXY = (n-1)*s_X*s_Y*r_XY

beta1 = SXY/SXX
beta0 = Ybar - beta1 * Xbar

R2  = r_XY^2
SSE = SYY*(1-R2)

sigma2 = SSE/(n-2)
sigma  = sqrt(sigma2)

SE_beta0 = sigma*sqrt(1/n + Xbar^2/((n-1)*s_X^2))
SE_beta1 = sigma*sqrt(           1/((n-1)*s_X^2))
```

Calculations "by hand" in R (continued)

```
# 95% CI for beta0
beta0 + c(-1, 1) * qt(.975, df = n-2) * SE_beta0

[1] 1.251761 1.483603

# 95% CI for beta1
beta1 + c(-1, 1) * qt(.975, df = n-2) * SE_beta1

[1] -0.044785794 -0.007962836

# pvalue for H0: beta0 <= 0 and P(beta0 > 0 | y)
pt(beta0 / SE_beta0, df = n - 2)

[1] 1

# pvalue for H0: beta1 <= 0 and P(beta1 > 0 | y)
pt(beta1 / SE_beta1, df = n - 2)

[1] 0.003102353

# pvalue for H0: beta1 = 0
2 * pt(-abs(beta1 / SE_beta1), df = n - 2)

[1] 0.006204706
```

Calculations by hand

$$
\begin{aligned}
SXX &= (n-1)s_x^2 = (39-1) \times 2.9354274^2 = 327.4358974 \\
SYY &= (n-1)s_Y^2 = (39-1) \times 0.1797731^2 = 1.2280974 \\
SXY &= (n-1)s_X s_Y r_{XY} = (39-1) \times 2.9354274 \times 0.1797731 \times -0.4306534 = -8.6358974 \\
\hat\beta_1 &= SXY/SXX = -8.6358974/327.4358974 = -0.0263743 \\
\hat\beta_0 &= \overline{Y} - \hat\beta_1 \overline{X} = 1.2202564 - (-0.0263743) \times 5.5897436 = 1.3676821 \\
R^2 &= r_{XY}^2 = (-0.4306534)^2 = 0.1854624 \\
SSE &= SYY(1-R^2) = 1.2280974(1-0.1854624) = 1.0003316 \\
\hat\sigma^2 &= SSE/(n-2) = 1.0003316/(39-2) = 0.027036 \\
\hat\sigma &= \sqrt{\hat\sigma^2} = \sqrt{0.027036} = 0.1644262 \\
SE(\hat\beta_0) &= \hat\sigma\sqrt{\frac{1}{n} + \frac{\overline{X}^2}{(n-1)s_x^2}} = 0.1644262\sqrt{\frac{1}{39} + \frac{5.5897436^2}{(39-1)*2.9354274^2}} = 0.0572111 \\
SE(\hat\beta_1) &= \hat\sigma\sqrt{\frac{1}{(n-1)s_x^2}} = 0.1644262\sqrt{\frac{1}{(39-1)*2.9354274^2}} = 0.0090867 \\
p_{H_A:\beta_0\neq 0} &= 2P\left(T_{n-2} < -\left|\frac{\hat\beta_0}{SE(\hat\beta_0)}\right|\right) = 2P(t_{37} < -23.9058799) = 4.2740348 \times 10^{-24} \\
p_{H_A:\beta_1\neq 0} &= 2P\left(T_{n-2} < -\left|\frac{\hat\beta_1}{SE(\hat\beta_1)}\right|\right) = 2P(t_{37} < -2.9025065) = 0.0062047 \\
CI_{95\%\ \beta_0} &= \hat\beta_0 \pm t_{n-2,1-a/2} SE(\hat\beta_0) \\
&= 1.3676821 \pm 2.0261925 \times 0.0572111 = (1.2517613, 1.4836028) \\
CI_{95\%\ \beta_1} &= \hat\beta_1 \pm t_{n-2,1-a/2} SE(\hat\beta_1) \\
&= -0.0263743 \pm 2.0261925 \times 0.0090867 = (-0.0447858, -0.0079628)
\end{aligned}
$$

Regression in R

```
m = lm(telomere.length ~ years, Telomeres)
summary(m)

Call:
lm(formula = telomere.length ~ years, data = Telomeres)

Residuals:
     Min       1Q   Median       3Q      Max
-0.42218 -0.08537  0.02056  0.10738  0.28869

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  1.367682   0.057211  23.906   <2e-16 ***
years       -0.026374   0.009087  -2.903   0.0062 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.1644 on 37 degrees of freedom
Multiple R-squared:  0.1855,Adjusted R-squared:  0.1634
F-statistic: 8.425 on 1 and 37 DF,  p-value: 0.006205

confint(m)

                  2.5 %       97.5 %
(Intercept)  1.25176134  1.483602799
years       -0.04478579 -0.007962836
```

Conclusion

Telomere ratio at the time of diagnosis of a child's chronic illness is estimated to be 1.37 with a 95% credible interval of (1.25, 1.48). For each year since diagnosis, the telomere ratio decreases on average by 0.026 with a 95% credible interval of (0.008, 0.045) . The proportion of variability in telomere length described by a linear regression on years since diagnosis is 18.5%.

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> *The correlation between chronicity of caregiving and mean telomere length is* -0.445 *(P <0.01). [*$R^2 = 0.198$ *was shown in the plot.]*

Remark I'm guessing our analysis and that reported in the paper don't match exactly due to a discrepancy in the data.

Summary

- The simple linear regression model is

$$Y_i \stackrel{ind}{\sim} N(\beta_0 + \beta_1 X_i, \sigma^2)$$

  where Y_i and X_i are the response and explanatory variable, respectively, for individual i.

- Know how to use R to obtain $\hat{\beta}_0$, $\hat{\beta}_1$, $\hat{\sigma}^2$, R^2, p-values, CIs, etc.

- Interpret regression output:
  - β_0 is the expected value for the response when the explanatory variable is 0.
  - β_1 is the expected increase in the response for each unit increase in the explanatory variable.
  - σ is the standard deviation of responses around their mean.
  - R^2 is the proportion of the total variation of the response variable explained by the model.