R03 - Regression: using logarithms

STAT 5870 (Engineering)
Iowa State University

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Parameter interpretation in regression

If

$$E[Y|X] = \beta_0 + \beta_1 X,$$

then

- β_0 is the expected response when X is zero and
- $d\beta_1$ is the expected (additive) increase in the response for a d unit (additive) increase in the explanatory variable.

For the following discussion,

- Y is always going to be the original response and
- X is always going to be the original explanatory variable.

Corn yield example

Suppose

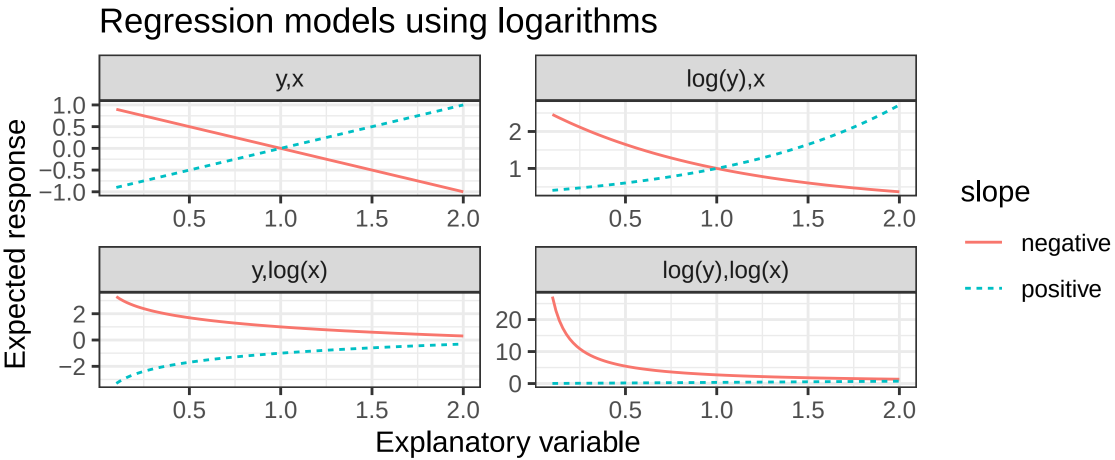
- Y is corn yield (bushels/acre)
- X is fertilizer level in lbs/acre

Then, if

$$E[Y|X] = \beta_0 + \beta_1 X$$

- β_0 is the expected corn yield (bushels/acre) when fertilizer level is zero and
- $d\beta_1$ is the expected increase in corn yield (bushels/acre) when fertilizer is increased by d lbs/acre.

Regression with logarithms (plotted on the original scale)



Response is logged

If

$$E[\log(Y)|X] = \beta_0 + \beta_1 X,$$

then we have

$$\text{Median}[Y|X] = e^{\beta_0+\beta_1 X} = e^{\beta_0} e^{\beta_1 X}$$

then

- e^{β_0} is the median of Y when X is zero
- $e^{d\beta_1}$ is the multiplicative increase in the median of Y for a d unit (additive) increase in the explanatory variable.

Response is logged

Let be Y is corn yield (bushels/acre) and X is fertilizer level in lbs/acre.
If we assume

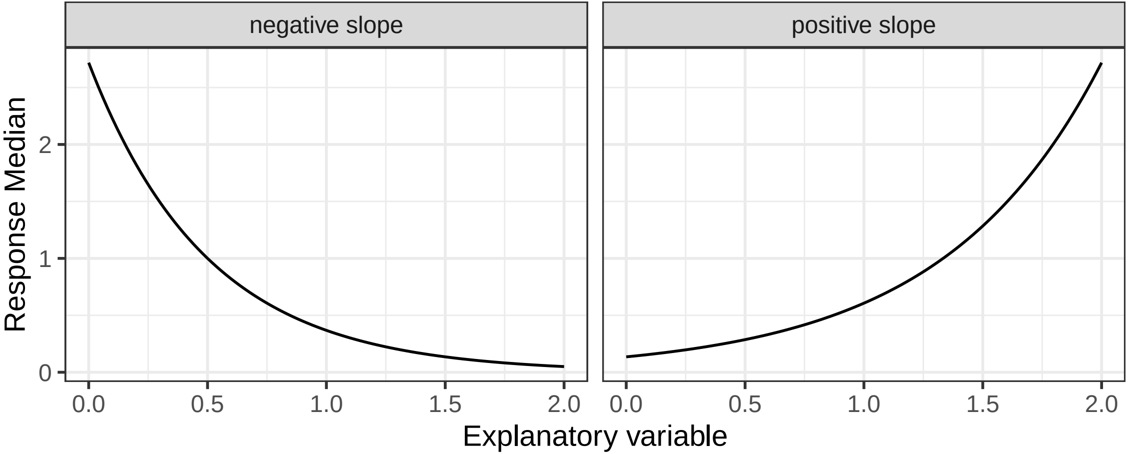
$$E[\log(Y)|X] = \beta_0 + \beta_1 X$$

then

$$\text{Median}[Y|X] = e^{\beta_0} e^{\beta_1 X}$$

- e^{β_0} is the median corn yield (bushels/acre) when fertilizer level is 0 (lbs/acre) and
- $e^{d\beta_1}$ is the multiplicative increase in median corn yield (bushels/acre) when fertilizer is increased by d lbs/acre.

Response is logged



Explanatory variable is logged

If

$$E[Y|X] = \beta_0 + \beta_1 \log(X),$$

then,

- β_0 is the expected response when X is 1 and
- $\beta_1 \log(d)$ is the expected (additive) increase in the response when X increases multiplicatively by d,e.g.
    - $\beta_1 \log(2)$ is the expected (additive) increase in the response for each doubling of X or
    - $\beta_1 \log(10)$ is the expected (additive) increase in the response for each ten-fold increase in X.

Explanatory variable is logged

Suppose

- Y is corn yield (bushels/acre)
- X is fertilizer level in lbs/acre

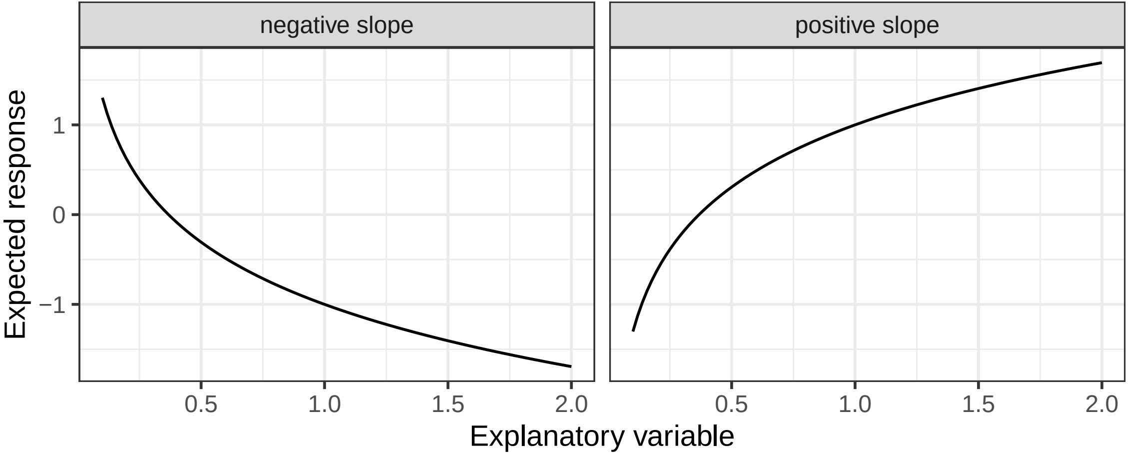
If

$$E[Y|X] = \beta_0 + \beta_1 \log(X)$$

then

- β_0 is the expected corn yield (bushels/acre) when fertilizer level is 1 lb/acre and
- $\beta_1 \log(2)$ is the expected (additive) increase in corn yield when fertilizer level is doubled.

Explanatory variable is logged



Both response and explanatory variable are logged

If

$$E[\log(Y)|X] = \beta_0 + \beta_1 \log(X),$$

then

$$\text{Median}[Y|X] = e^{\beta_0} X^{\beta_1},$$

and thus

- e^{β_0} is the median of Y when X is 1 and
- d^{β_1} is the multiplicative increase in the median of the response when X increases multiplicatively by d, e.g.
  - 2^{β_1} is the multiplicative increase in the median of the response for each doubling of X or
  - 10^{β_1} is the multiplicative increase in the median of the response for each ten-fold increase in X.

Both response and explanatory variables are logged

Suppose

- Y is corn yield (bushels/acre)
- X is fertilizer level in lbs/acre

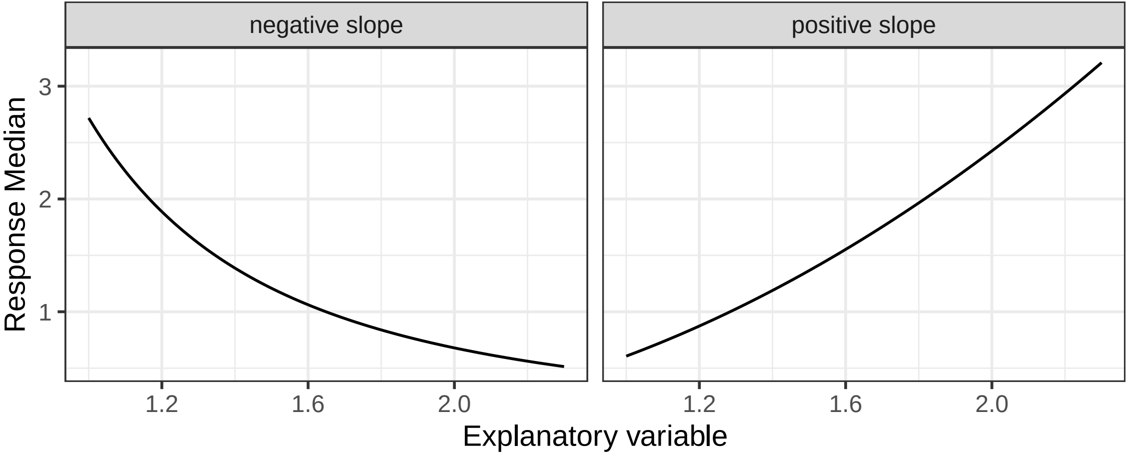
If

$$E[\log(Y)|X] = \beta_0 + \beta_1 \log(X) \quad \text{or} \quad \text{Median}[Y|X] = e^{\beta_0} e^{\beta_1 \log(X)} = e^{\beta_0} X^{\beta_1},$$

then

- e^{β_0} is the median corn yield (bushels/acre) at 1 lb/acre of fertilizer and
- 2^{β_1} is the multiplicative increase in median corn yield (bushels/acre) when fertilizer is doubled.

Both response and explanatory variables are logged



Why use logarithms

The most common transformation of either the response or explanatory variable(s) is to take logarithms because

- linearity will often then be approximately true,
- the variance will likely be approximately constant,
- influence of some observations may decrease, and
- there is a (relatively) convenient interpretation.

Summary of interpretations when using logarithms

- When using the log of the response,
  - β_0 determines the median response
  - β_1 determines the multiplicative increase in the median response
- When using the log of the explanatory variable (X),
  - β_0 determines the response when $X = 1$
  - β_1 determines the increase in the response when there is a multiplicative increase in X

Constructing credible intervals

Recall the model

$$Y_i \stackrel{ind}{\sim} N(\beta_0 + \beta_1 X_i, \sigma^2).$$

Let (L, U) be a $100(1-a)\%$ credible interval for β.

For ease of interpretation, it is often convenient to calculate functions of β, e.g.

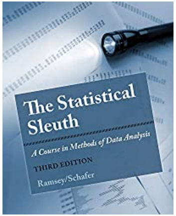
$$f(\beta) = d\beta \qquad \text{and} \qquad f(\beta) = e^{\beta}.$$

A $100(1-a)\%$ credible interval for $f(\beta)$ (when f is monotonic) is

$$(f(L), f(U)).$$

Breakdown times

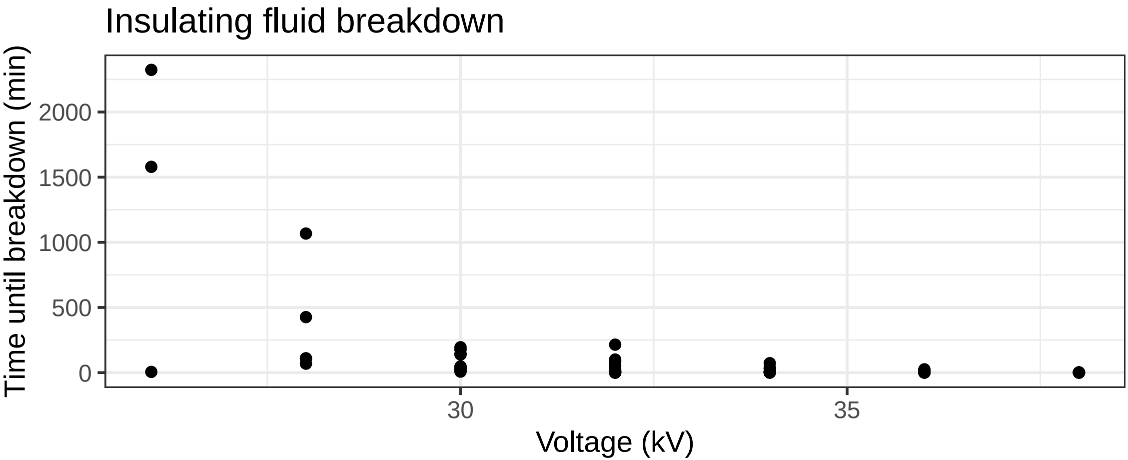
In an industrial laboratory, under uniform conditions, batches of electrical insulating fluid were subjected to constant voltages (kV) until the insulating property of the fluids broke down. Seven different voltage levels were studied and the measured responses were the times (minutes) until breakdown.



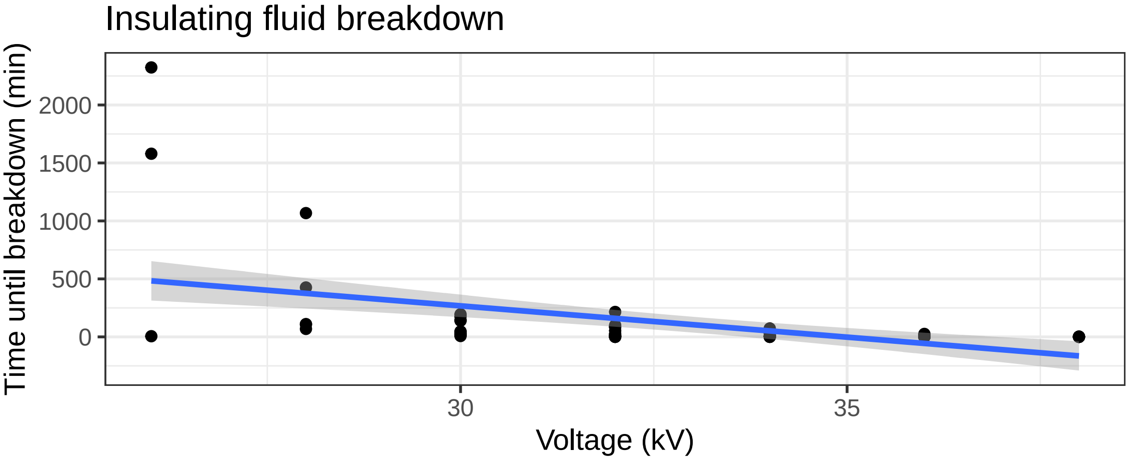

```
summary(Sleuth3::case0802)
```

```
      Time              Voltage            Group   
 Min.    :   0.090   Min.    :26.00   Group1: 3  
 1st Qu.:   1.617   1st Qu.:31.50   Group2: 5  
 Median :   6.925   Median :34.00   Group3:11  
 Mean   :  98.558   Mean    :33.13   Group4:15  
 3rd Qu.:  38.383   3rd Qu.:36.00   Group5:19  
 Max.    :2323.700   Max.    :38.00   Group6:15  
                                      Group7: 8  
```

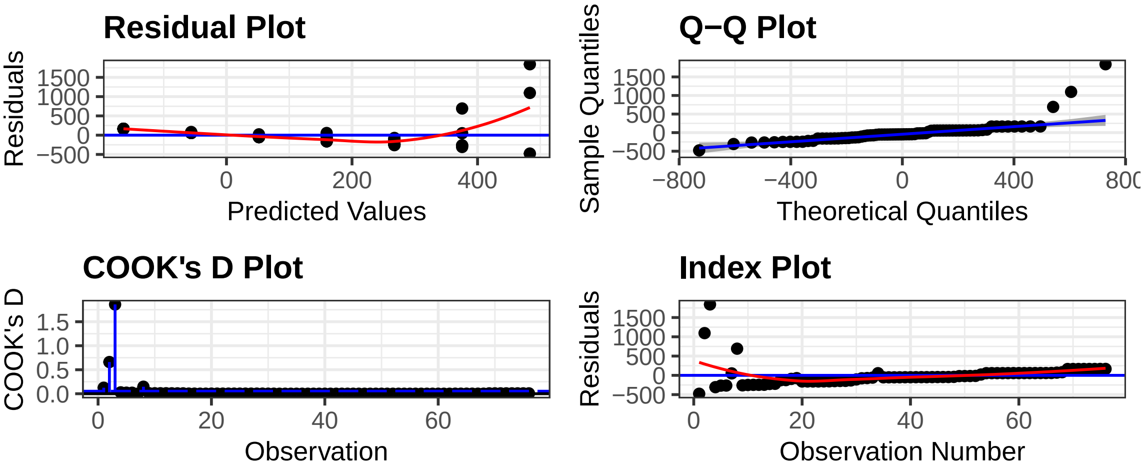
Insulating fluid breakdown



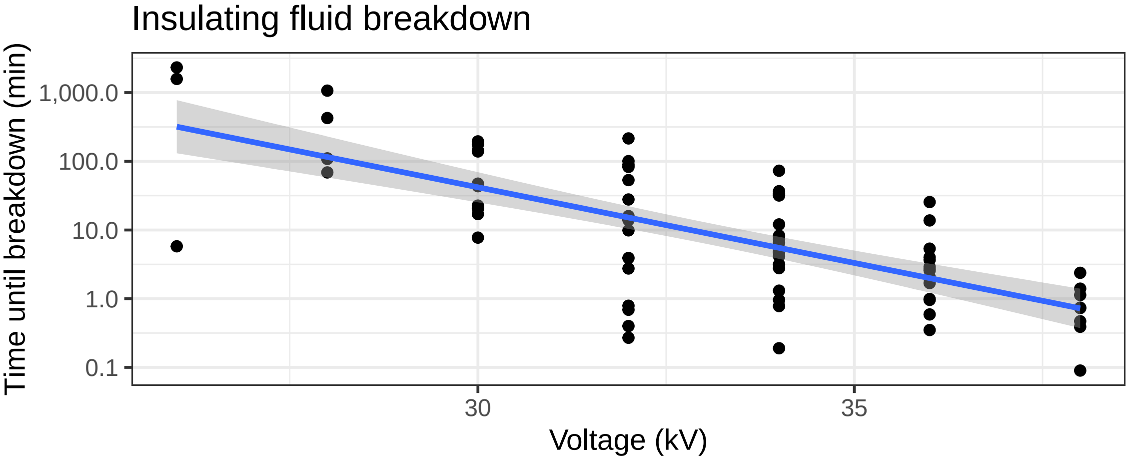
Insulating fluid breakdown



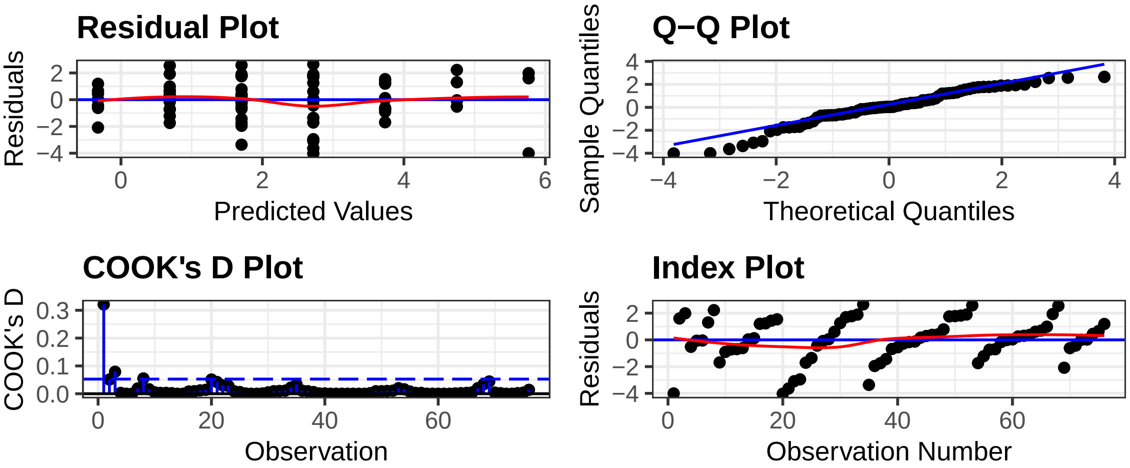
Run the regression and look at diagnostics



Logarithm of time (response)



Logarithm of time (response): residuals



Summary

```
m <- lm(log(Time) ~ I(Voltage-30), Sleuth3::case0802)
exp(m$coefficients)

    (Intercept) I(Voltage - 30)
       41.86752         0.60208

exp(confint(m))

                     2.5 %     97.5 %
(Intercept)      25.2582342 69.3987157
I(Voltage - 30)   0.5370152  0.6750281
```

- At 30 kV, the median breakdown time is estimated to be 42 minutes with a 95% credible interval of (25, 69).
- Each 1 kV increase in voltage was associated with a 40% (32%, 46%) reduction in median breakdown time.