R04 - Regression with Categorical Explanatory Variables

STAT 5870 (Engineering)
Iowa State University

November 11, 2024

Binary explanatory variable

Recall the simple linear regression model

$$Y_i \stackrel{ind}{\sim} N(\beta_0 + \beta_1 X_i, \sigma^2).$$

If we have a binary explanatory variable, i.e. the explanatory variable only has two levels say level A and level B, we can code it as

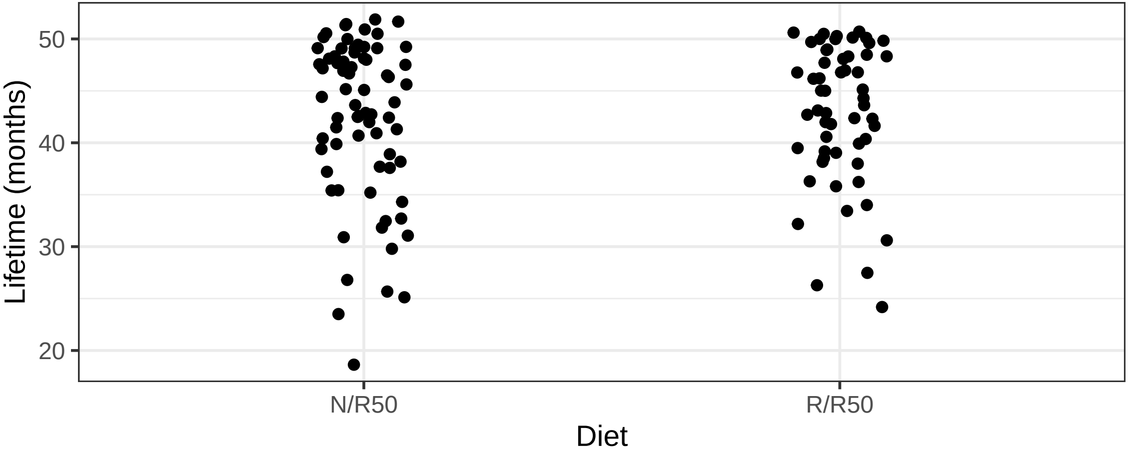
$$X_i = \mathrm{I}(\text{observation } i \text{ is level A})$$

where $\mathrm{I}(statement)$ is an indicator function that is 1 when $statement$ is true and 0 otherwise. Then

- β_0 is the expected response for level B,
- $\beta_0 + \beta_1$ is the expected response for level A, and
- β_1 is the expected difference in response (level A minus level B).

Mice lifetimes

```
Sleuth3::case0501
```



Regression model for mice lifetimes

Let

$$Y_i \stackrel{ind}{\sim} N(\beta_0 + \beta_1 X_i, \sigma^2)$$

where Y_i is the lifetime of the ith mouse and

$$X_i = \mathrm{I}(Diet_i = \text{N/R50})$$

then

$$\begin{aligned} E[\text{Lifetime}|\text{R/R50}] &= E[Y_i|X_i = 0] &= \beta_0 \\ E[\text{Lifetime}|\text{N/R50}] &= E[Y_i|X_i = 1] &= \beta_0 + \beta_1 \end{aligned}$$

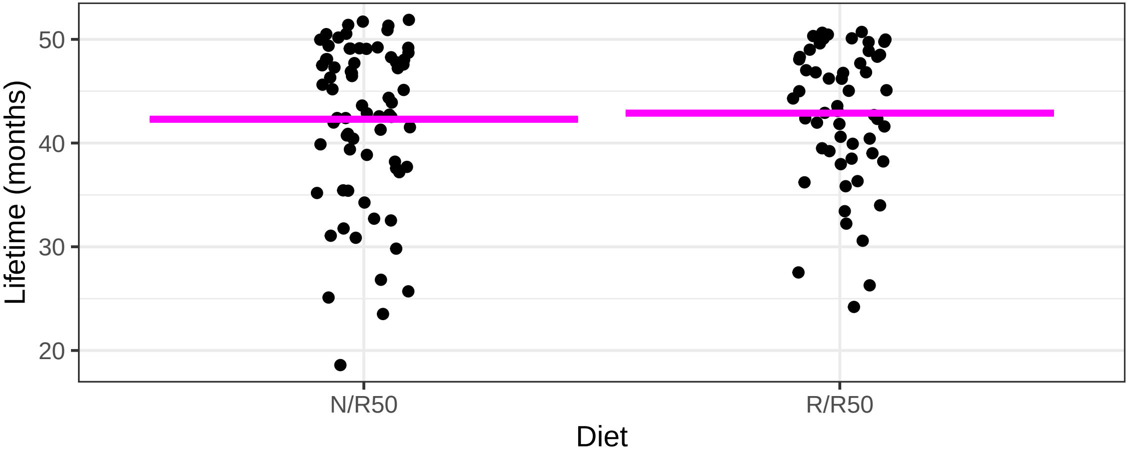
and

$$\begin{aligned} &E[\text{Lifetime difference}] \\ &\quad = E[\text{Lifetime}|\text{N/R50}] - E[\text{Lifetime}|\text{R/R50}] \\ &\quad = (\beta_0 + \beta_1) - \beta_0 = \beta_1. \end{aligned}$$

R code

```
case0501$X <- ifelse(case0501$Diet == "N/R50", 1, 0)
(m <- lm(Lifetime ~ X, data = case0501, subset = Diet %in% c("R/R50","N/R50")))

Call:
lm(formula = Lifetime ~ X, data = case0501, subset = Diet %in%
    c("R/R50", "N/R50"))

Coefficients:
(Intercept)            X
    42.8857      -0.5885

confint(m)

                2.5 %    97.5 %
(Intercept) 40.952257 44.819172
X           -3.174405  1.997342

predict(m, data.frame(X=1), interval = "confidence") # Expected lifetime on N/R50

       fit      lwr     upr
1 42.29718 40.58007 44.0143
```

Mice lifetimes



Equivalence to a two-sample t-test

Recall that our two-sample t-test had the model

$$Y_{ij} \stackrel{ind}{\sim} N(\mu_j, \sigma^2)$$

for groups $j = 0, 1$. This is equivalent to our current regression model where

$$\begin{aligned} \mu_0 &= \beta_0 \\ \mu_1 &= \beta_0 + \beta_1 \end{aligned}$$

assuming

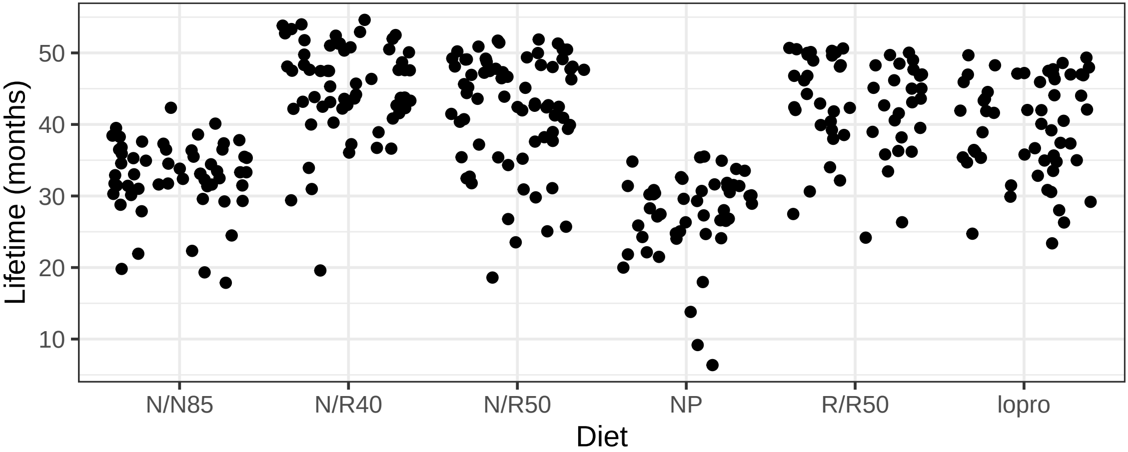
- μ_0 represents the mean for the R/R50 group and
- μ_1 represents the mean for N/R50 group.

When the models are effectively the same, but have different parameters we say the model is reparameterized.

Equivalence

```
summary(m)$coefficients[2,4] # p-value

[1] 0.6531748

confint(m)

                2.5 %    97.5 %
(Intercept) 40.952257 44.819172
X           -3.174405  1.997342

t.test(Lifetime ~ Diet, data = case0501, subset = Diet %in% c("R/R50","N/R50"), var.equal=TRUE)

Two Sample t-test

data:  Lifetime by Diet
t = -0.45044, df = 125, p-value = 0.6532
alternative hypothesis: true difference in means between group N/R50 and group R/R50 is not equal to 0
95 percent confidence interval:
 -3.174405  1.997342
sample estimates:
mean in group N/R50 mean in group R/R50
           42.29718            42.88571
```

Using a categorical variable as an explanatory variable.



Regression with a categorical variable

1. Choose one of the levels as the reference level, e.g. N/N85
2. Construct dummy variables using indicator functions, i.e.

$$\mathrm{I}(A) = \begin{cases} 1 & A \text{ is TRUE} \\ 0 & A \text{ is FALSE} \end{cases}$$

   for the other levels, e.g.

$$\begin{aligned} X_{i,1} &= \mathrm{I}(\text{diet for observation } i \text{ is N/R40}) \\ X_{i,2} &= \mathrm{I}(\text{diet for observation } i \text{ is N/R50}) \\ X_{i,3} &= \mathrm{I}(\text{diet for observation } i \text{ is NP}) \\ X_{i,4} &= \mathrm{I}(\text{diet for observation } i \text{ is R/R50}) \\ X_{i,5} &= \mathrm{I}(\text{diet for observation } i \text{ is lopro}) \end{aligned}$$

3. Estimate the parameters of a multiple regression model using these dummy variables.

Regression model

Our regression model becomes

$$Y_i \stackrel{ind}{\sim} N(\beta_0 + \beta_1 X_{i,1} + \beta_2 X_{i,2} + \beta_3 X_{i,3} + \beta_4 X_{i,4} + \beta_5 X_{i,5}, \sigma^2)$$

where

- β_0 is the expected lifetime for the N/N85 group
- $\beta_0 + \beta_1$ is the expected lifetime for the N/R40 group
- $\beta_0 + \beta_2$ is the expected lifetime for the N/R50 group
- $\beta_0 + \beta_3$ is the expected lifetime for the NP group
- $\beta_0 + \beta_4$ is the expected lifetime for the R/R50 group
- $\beta_0 + \beta_5$ is the expected lifetime for the lopro group

and thus β_p for $p > 0$ is the difference in expected lifetimes between one group and a reference group.

R code

```
case0501 <- case0501 |>
  mutate(X1 = Diet == "N/R40",
         X2 = Diet == "N/R50",
         X3 = Diet == "NP",
         X4 = Diet == "R/R50",
         X5 = Diet == "lopro")

m <- lm(Lifetime ~ X1 + X2 + X3 + X4 + X5, data = case0501)
m


Call:
lm(formula = Lifetime ~ X1 + X2 + X3 + X4 + X5, data = case0501)

Coefficients:
(Intercept)       X1TRUE       X2TRUE       X3TRUE       X4TRUE       X5TRUE
     32.691       12.425        9.606       -5.289       10.194        6.994

confint(m)


                2.5 %    97.5 %
(Intercept) 30.951394 34.431062
X1TRUE       9.995893 14.854984
X2TRUE       7.269897 11.942013
X3TRUE      -7.848142 -2.730232
X4TRUE       7.723030 12.665943
X5TRUE       4.523030  9.465943
```

R code (cont.)

```
summary(m)

Call:
lm(formula = Lifetime ~ X1 + X2 + X3 + X4 + X5, data = case0501)

Residuals:
     Min       1Q   Median       3Q      Max
-25.5167  -3.3857   0.8143   5.1833  10.0143

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  32.6912     0.8846  36.958  < 2e-16 ***
X1TRUE       12.4254     1.2352  10.059  < 2e-16 ***
X2TRUE        9.6060     1.1877   8.088 1.06e-14 ***
X3TRUE       -5.2892     1.3010  -4.065 5.95e-05 ***
X4TRUE       10.1945     1.2565   8.113 8.88e-15 ***
X5TRUE        6.9945     1.2565   5.567 5.25e-08 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 6.678 on 343 degrees of freedom
Multiple R-squared:  0.4543,Adjusted R-squared:  0.4463
F-statistic:  57.1 on 5 and 343 DF,  p-value: < 2.2e-16
```

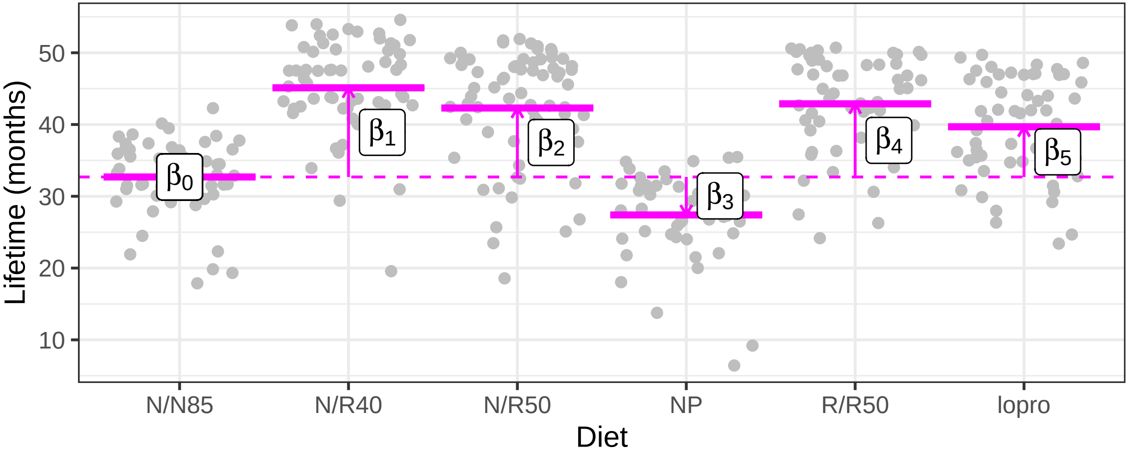
Interpretation

- $\beta_0 = E[Y_i|\text{reference level}]$, i.e. expected response for the reference level
  Note: the only way $X_{i,1} = \cdots = X_{i,p} = 0$ is if all indicators are zero, i.e. at the reference level.
- $\beta_p, p > 0$: expected change in the response moving from the reference level to the level associated with the p^{th} dummy variable
  Note: the only way for $X_{i,p}$ to increase by one is if initially $X_{i,1} = \cdots = X_{i,p} = 0$ and now $X_{i,p} = 1$

For example,

- The expected lifetime for mice on the N/N85 diet is 32.7 (31.0,34.4) months.
- The expected increase in lifetime for mice on the N/R40 diet compared to the N/N85 diet is 12.4 (10.0,14.9) months.
- The model explains 45% of the variability in mice lifetimes.

Using a categorical variable as an explanatory variable.



Equivalence to multiple group model

Recall that we had a multiple group model

$$Y_{ij} \stackrel{ind}{\sim} N(\mu_j, \sigma^2)$$

for groups $j = 0, 1, 2, \ldots, 5$.

Our regression model is a reparameterization of the multiple group model:

$$\begin{array}{lll} N/N85: & \mu_0 & = \beta_0 \\ N/R40: & \mu_1 & = \beta_0 + \beta_1 \\ N/R50: & \mu_2 & = \beta_0 + \beta_2 \\ NP: & \mu_3 & = \beta_0 + \beta_3 \\ R/R50: & \mu_4 & = \beta_0 + \beta_4 \\ lopro: & \mu_5 & = \beta_0 + \beta_5 \end{array}$$

assuming the groups are labeled appropriately.

Summary

1. Choose one of the levels as the reference level.
2. Construct dummy variables using indicator functions for all other levels, e.g.

$$X_i = \mathrm{I}(\text{observation } i \text{ is <some non-reference level>}).$$

3. Estimate the parameters of a multiple regression model using these dummy variables.