R06a - Interpreting Regression p-values as Posterior Probabilities

STAT 5870 (Engineering)
Iowa State University

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Regression p-values

Recall the regression model

$$Y_i \stackrel{ind}{\sim} N(\mu_i, \sigma^2), \qquad \mu_i = \beta_0 + \beta_1 X_{i,1} + \cdots + \beta_p X_{i,p}$$

A common hypothesis test is

$$H_0 : \beta_j = 0 \qquad \text{versus} \qquad H_A : \beta_j \neq 0$$

which has

$$p\text{-value} = 2P\left(T > |t|\right)$$

where $T \sim t_{n-(p+1)}$ and $t = \hat{\beta}_j / SE(\beta_j)$.

Example Regression Output

```
Call:
lm(formula = Speed ~ Conditions * log(NetToWinner), data = Sleuth3::ex0920)

Residuals:
     Min       1Q   Median       3Q      Max 
-1.50551 -0.32127 -0.00219  0.35201  1.13026 

Coefficients:
                                Estimate Std. Error t value Pr(>|t|)    
(Intercept)                     33.23367    0.34584  96.095  < 2e-16 ***
ConditionsSlow                  -2.04517    0.72404  -2.825   0.0056 ** 
log(NetToWinner)                 0.27830    0.02942   9.458 5.88e-16 ***
ConditionsSlow:log(NetToWinner)  0.08664    0.06583   1.316   0.1908    
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.4978 on 112 degrees of freedom
Multiple R-squared:  0.7015,Adjusted R-squared:  0.6935 
F-statistic: 87.75 on 3 and 112 DF,  p-value: < 2.2e-16
```

Bayesian Posterior Probabilities

With prior $p(\beta, \sigma^2) \propto 1/\sigma^2$, we have

$$\beta_j|y \sim t_{n-(p+1)}\left(\hat{\beta}_j, SE(\beta_j)^2\right).$$

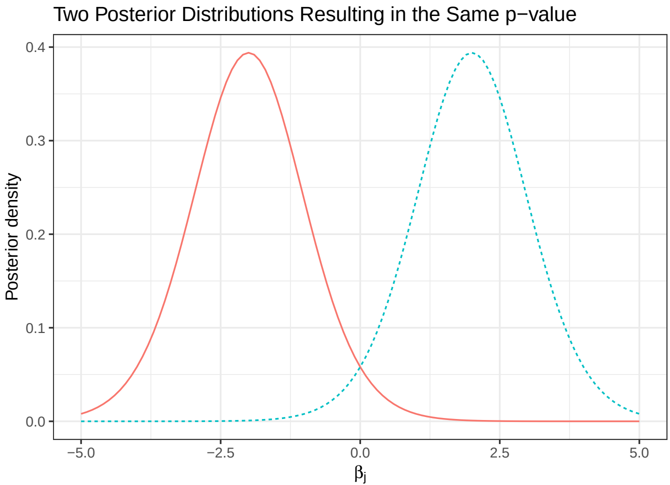
Thus

$$P\left(\beta_j > 0|\, y\right) = P\left(\left.\frac{\beta_j - \hat{\beta}_j}{SE(\beta_j)} > \frac{0 - \hat{\beta}_j}{SE(\beta_j)}\right| y\right) = P\left(T > -t\right)$$

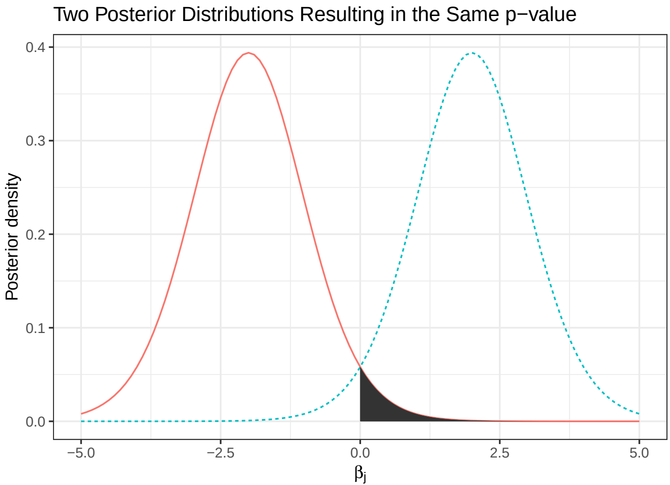
which is very close to

$$p\text{-value} = 2P\left(T > |t|\right).$$

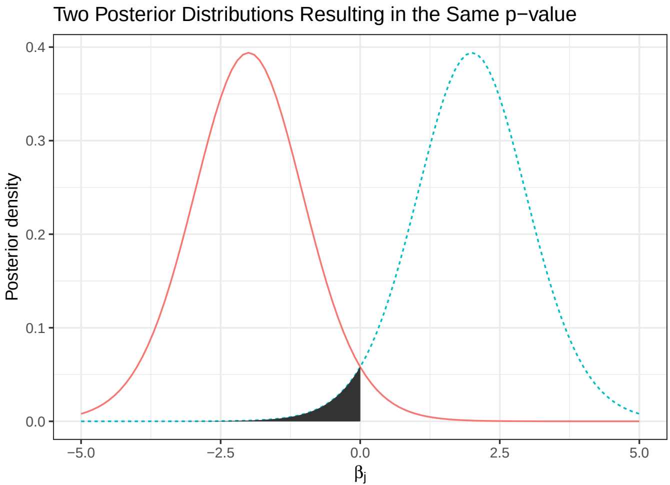
Visualizing Posterior Distribution



Visualizing Posterior Distribution



Visualizing Posterior Distribution



Interpreting Regression p-values as Posterior Probabilities

Suppose we have a p-value for $H_0 : \beta_j = 0$ vs $H_A : \beta_j \neq 0$. Then

- If $\hat{\beta}_j < 0$, then
$$P(\beta_j > 0|y) = p\text{-value}/2.$$
- If $\hat{\beta}_j > 0$, then
$$P(\beta_j < 0|y) = p\text{-value}/2.$$

Alternatively,

- If $\hat{\beta}_j < 0$, then
$$P(\beta_j < 0|y) = 1 - p\text{-value}/2.$$
- If $\hat{\beta}_j > 0$, then
$$P(\beta_j > 0|y) = 1 - p\text{-value}/2.$$

Example Interpretation

```
                            Estimate Std. Error t value Pr(>|t|)
(Intercept)                    33.23       0.35   96.09     0.00
ConditionsSlow                 -2.05       0.72   -2.82     0.01
log(NetToWinner)                0.28       0.03    9.46     0.00
ConditionsSlow:log(NetToWinner) 0.09       0.07    1.32     0.19
```

Intercept	$P(\beta_0 > 0 \mid y) \approx 1$
ConditionsSlow	$P(\beta_1 < 0 \mid y) \approx 0.99$
log(NetToWinner)	$P(\beta_2 > 0 \mid y) \approx 1$
ConditionsSlow:log(NetToWinner)	$P(\beta_3 > 0 \mid y) \approx 0.90$

Summary

Suppose we have a regression p-value for $H_0 : \beta_j = 0$ vs $H_A : \beta_j \neq 0$. Then

- If $\hat{\beta}_j < 0$, then
$$P(\beta_j < 0|y) = 1 - p\text{-value}/2.$$
- If $\hat{\beta}_j > 0$, then
$$P(\beta_j > 0|y) = 1 - p\text{-value}/2.$$