R07 - Contrasts

STAT 5870 (Engineering)
Iowa State University

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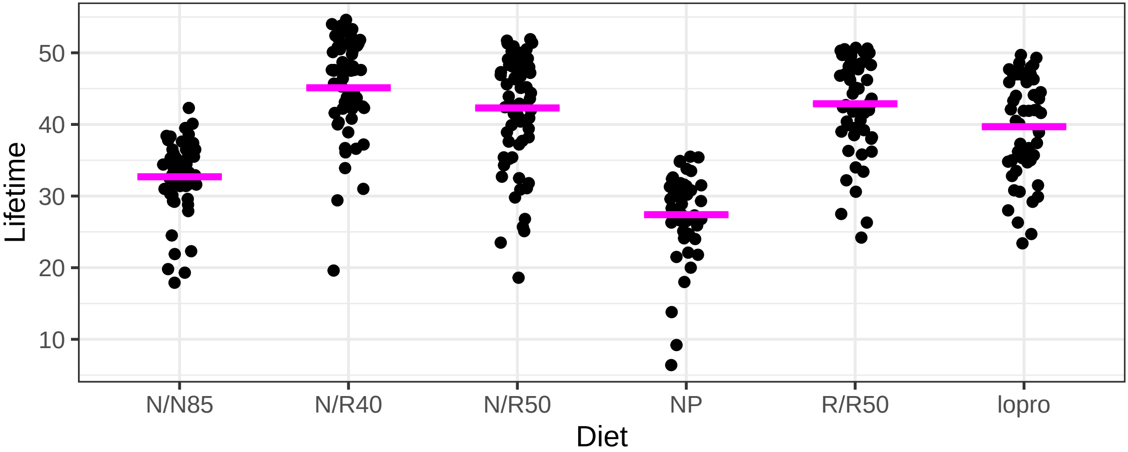
Diet Effect on Mice Lifetimes



ANOVA and Regression Models

ANOVA model:

$$Y_{ij} \stackrel{ind}{\sim} N(\mu_j, \sigma^2)$$

with Y_{ij} being the lifetime for the ith mouse on the jth diet for $j = 0, 1, 2, 3, 4, 5$.

Regression model:

$$Y_i \stackrel{ind}{\sim} N(\beta_0 + \beta_1 X_{i,1} + \ldots + \beta_p X_{i,p}, \sigma^2)$$

where Y_i is the lifetime for the ith mouse and $X_{i,j}$ is an indicator for the ith mouse being on the jth diet.

Reparameterized model since

$$\mu_0 = \beta_0 \quad \text{and} \quad \mu_j = \beta_0 + \beta_j$$

for $j > 0$.

Scientific questions

Here are a few example scientific questions:

1. What is the effect of pre-wean calorie restriction on mean lifetimes?
2. What is the difference in mean lifetimes between mice on a 40 kcal diet compared to those on a 50 kcal diet?
3. What is the effect of high calorie vs low calorie diets on mean lifetimes?

We can compute contrasts:

$$
\begin{aligned}
\gamma_1 = &\quad \mu_{R/R50} - \mu_{N/R50} \\
\gamma_2 = &\quad \mu_{N/R40} - \tfrac{1}{2}(\mu_{N/R50} + \mu_{R/R50}) \\
\gamma_3 = &\quad \tfrac{1}{4}(\mu_{N/R50} + \mu_{R/R50} + \mu_{N/R40} + \mu_{lopro}) \\
&\quad -\tfrac{1}{2}(\mu_{NP} + \mu_{N/N85})
\end{aligned}
$$

Contrasts

A linear combination of group means has the form

$$\gamma = C_1\mu_1 + C_2\mu_2 + \ldots + C_J\mu_J$$

where C_j are known coefficients and μ_j are the unknown population means.

A linear combination with $C_1 + C_2 + \cdots + C_J = 0$ is a contrast.

Contrast interpretation is usually best if $|C_1| + |C_2| + \cdots + |C_J| = 2$, i.e. the positive coefficients sum to 1 and the negative coefficients sum to -1.

Inference on Contrasts

Contrast
$$\gamma = C_1\mu_1 + C_2\mu_2 + \cdots + C_J\mu_J \quad \text{with} \quad \hat{\gamma} = C_1\overline{Y}_1 + C_2\overline{Y}_2 + \cdots + C_J\overline{Y}_J$$
with standard error
$$SE(\hat{\gamma}) = \hat{\sigma}\sqrt{\frac{C_1^2}{n_1} + \frac{C_2^2}{n_2} + \cdots + \frac{C_J^2}{n_J}}.$$
p-values for $H_0 : \gamma = g_0$ vs $H_A : \gamma \neq g_0$ and posterior probabilities (i.e. $2P(\gamma > 0|y)$ or $2P(\gamma < 0|y)$):
$$t = \frac{g - g_0}{SE(g)}, \quad p = 2P(T_{n-J} < -|t|).$$
Two-sided equal-tail $100(1-\alpha)\%$ confidence/credible intervals:
$$g \pm t_{n-J,1-\alpha/2} SE(g).$$

Contrasts for mice lifetime dataset

For these contrasts:

1. Difference in mean lifetimes for N/R50 v R/R50 diet
2. Difference in mean lifetimes for N/R40 v N/R50 and R/R50 combined
3. Difference in mean lifetimes for high calorie (NP and N/N85) diets v low calorie (others) diets

$H_0 : \gamma = 0 \qquad H_A : \gamma \neq 0 :$

$$
\begin{aligned}
\gamma_1 = &\quad \mu_{R/R50} - \mu_{N/R50} \\
\gamma_2 = &\quad \mu_{N/R40} - \frac{1}{2}(\mu_{N/R50} + \mu_{R/R50}) \\
\gamma_3 = &\quad \frac{1}{4}(\mu_{N/R50} + \mu_{R/R50} + \mu_{N/R40} + \mu_{lopro}) \\
&\quad -\frac{1}{2}(\mu_{NP} + \mu_{N/N85})
\end{aligned}
$$

	N/N85	N/R40	N/R50	NP	R/R50	lopro
early rest - none @ 50kcal	0.00	0.00	-1.00	0.00	1.00	0.00
40kcal/week - 50kcal/week	0.00	1.00	-0.50	0.00	-0.50	0.00
lo cal - hi cal	-0.50	0.25	0.25	-0.50	0.25	0.25

Fit the Multiple Regression Model

```
m <- lm(Lifetime ~ Diet, data = Sleuth3::case0501)
summary(m)

Call:
lm(formula = Lifetime ~ Diet, data = Sleuth3::case0501)

Residuals:
     Min       1Q   Median       3Q      Max 
-25.5167  -3.3857   0.8143   5.1833  10.0143 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)  32.6912     0.8846  36.958  < 2e-16 ***
DietN/R40    12.4254     1.2352  10.059  < 2e-16 ***
DietN/R50     9.6060     1.1877   8.088 1.06e-14 ***
DietNP       -5.2892     1.3010  -4.065 5.95e-05 ***
DietR/R50    10.1945     1.2565   8.113 8.88e-15 ***
Dietlopro     6.9945     1.2565   5.567 5.25e-08 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 6.678 on 343 degrees of freedom
Multiple R-squared:  0.4543,Adjusted R-squared:  0.4463 
F-statistic:  57.1 on 5 and 343 DF,  p-value: < 2.2e-16
```

Estimate Group Means

```
library("emmeans")
em <- emmeans(m, ~ Diet)
em

 Diet   emmean    SE  df lower.CL upper.CL
 N/N85    32.7 0.885 343     31.0     34.4
 N/R40    45.1 0.862 343     43.4     46.8
 N/R50    42.3 0.793 343     40.7     43.9
 NP       27.4 0.954 343     25.5     29.3
 R/R50    42.9 0.892 343     41.1     44.6
 lopro    39.7 0.892 343     37.9     41.4

Confidence level used: 0.95
```

```
K_list

$`early rest - none @ 50kcal`
[1]  0  0 -1  0  1  0

$`40kcal/week - 50kcal/week`
[1]  0.0  1.0 -0.5  0.0 -0.5  0.0

$`lo cal - hi cal`
[1] -0.50  0.25  0.25 -0.50  0.25  0.25
```

```
co <- contrast(em, K_list)

# p-values (and posterior tail probabilities)
co

 contrast                   estimate   SE  df t.ratio p.value
 early rest - none @ 50kcal    0.589 1.19 343   0.493  0.6223
 40kcal/week - 50kcal/week     2.525 1.05 343   2.408  0.0166
 lo cal - hi cal              12.450 0.78 343  15.961  <.0001

# confidence/credible intervals
confint(co)

 contrast                   estimate   SE  df lower.CL upper.CL
 early rest - none @ 50kcal    0.589 1.19 343   -1.759     2.94
 40kcal/week - 50kcal/week     2.525 1.05 343    0.463     4.59
 lo cal - hi cal              12.450 0.78 343   10.915    13.98

Confidence level used: 0.95
```

Summary

- Contrasts are linear combinations of means where the coefficients sum to zero
- t-test tools are used to calculate pvalues and confidence intervals

Sulfur effect on scab disease in potatoes

The experiment was conducted to investigate the effect of sulfur on controlling scab disease in potatoes. There were seven treatments: control, plus spring and fall application of 300, 600, 1200 lbs/acre of sulfur. The response variable was percentage of the potato surface area covered with scab averaged over 100 random selected potatoes. A completely randomized design was used with 8 replications of the control and 4 replications of the other treatments.

Cochran and Cox. (1957) Experimental Design (2nd ed). pg96 and Agron. J. 80:712-718 (1988)

Scientific questions:

- Does sulfur have any impact at all?
- What is the difference between spring and fall application of sulfur?
- What is the effect of increased sulfur application?

Data

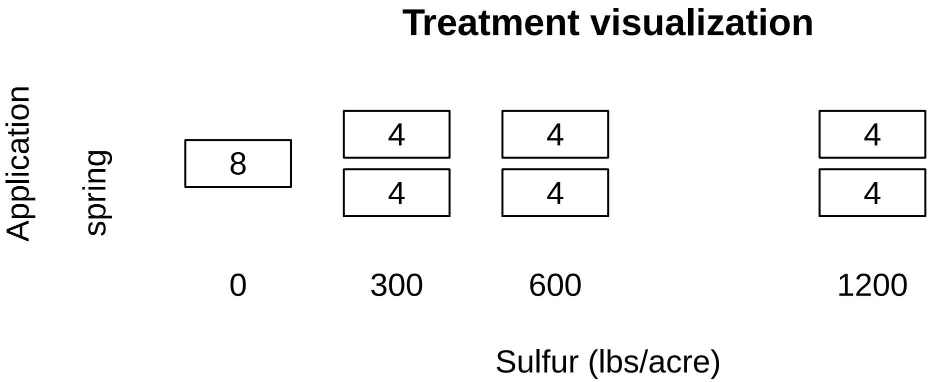
	inf	trt	row	col	sulfur	application	treatment
1	9	F3	4	1	300	fall	F3
2	12	0	4	2	0	not applicable	0
3	18	S6	4	3	600	spring	S6
4	10	F12	4	4	1200	fall	F12
5	24	S6	4	5	600	spring	S6
6	17	S12	4	6	1200	spring	S12
7	30	S3	4	7	300	spring	S3
8	16	F6	4	8	600	fall	F6
9	10	0	3	1	0	not applicable	0
10	7	S3	3	2	300	spring	S3
11	4	F12	3	3	1200	fall	F12
12	10	F6	3	4	600	fall	F6
13	21	S3	3	5	300	spring	S3
14	24	0	3	6	0	not applicable	0
15	29	0	3	7	0	not applicable	0
16	12	S6	3	8	600	spring	S6
17	9	F3	2	1	300	fall	F3
18	7	S12	2	2	1200	spring	S12
19	18	F6	2	3	600	fall	F6
20	30	0	2	4	0	not applicable	0
21	18	F6	2	5	600	fall	F6
22	16	S12	2	6	1200	spring	S12
23	16	F3	2	7	300	fall	F3
24	4	F12	2	8	1200	fall	F12
25	9	S3	1	1	300	spring	S3
26	18	0	1	2	0	not applicable	0
27	17	S12	1	3	1200	spring	S12

Design

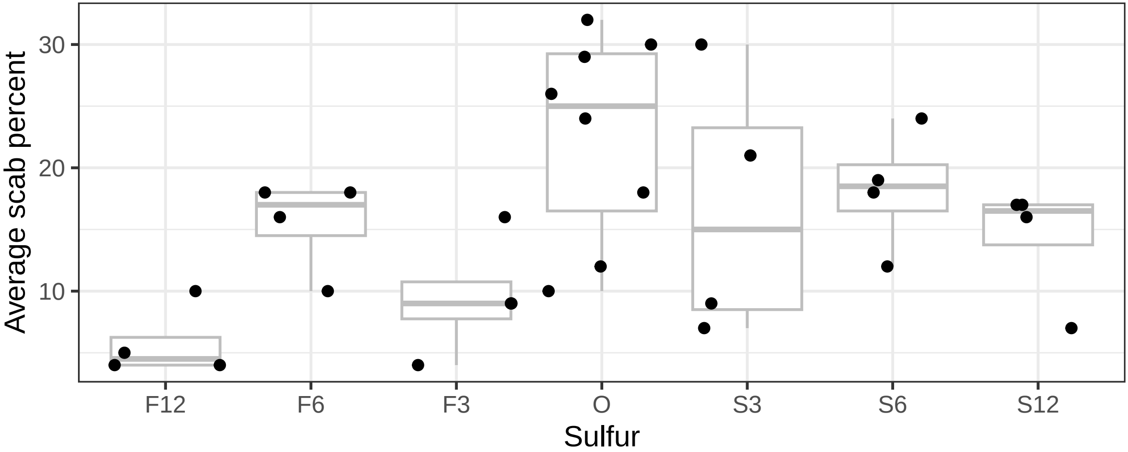
Completely randomized design
potato scab experiment

row								
	F3	O	S6	F12	S6	S12	S3	F6
3	O	S3	F12	F6	S3	O	O	S6
	F3	S12	F6	O	F6	S12	F3	F12
1	S3	O	S12	S6	O	F12	O	F3
col	1	2	3	4	5	6	7	8

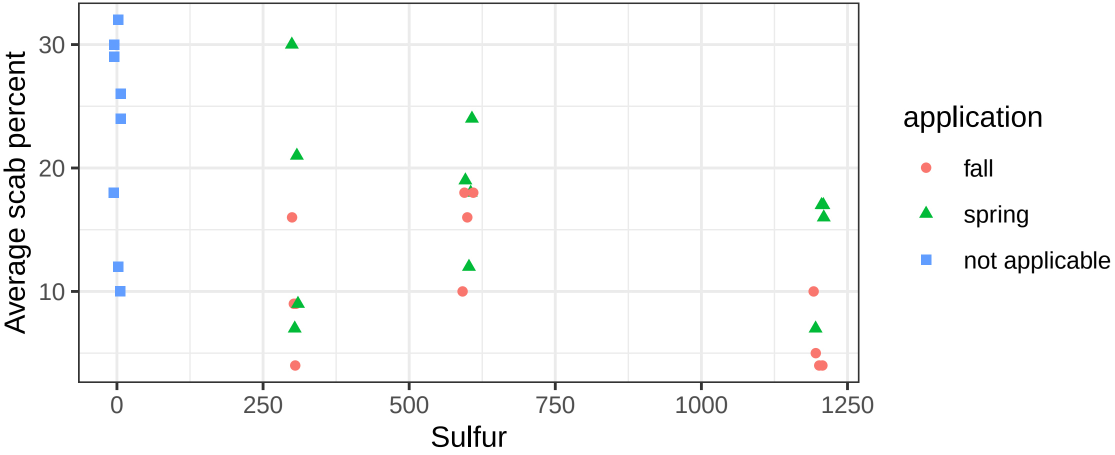
Design


Treatment visualization
Application
spring
8
4
4
4
4
4
4
0
300
600
1200
Sulfur (lbs/acre)

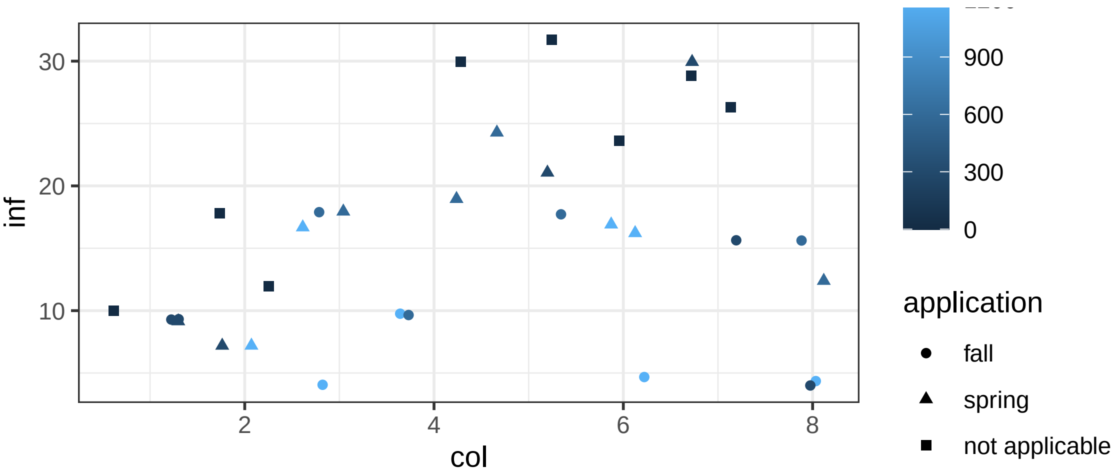
Data



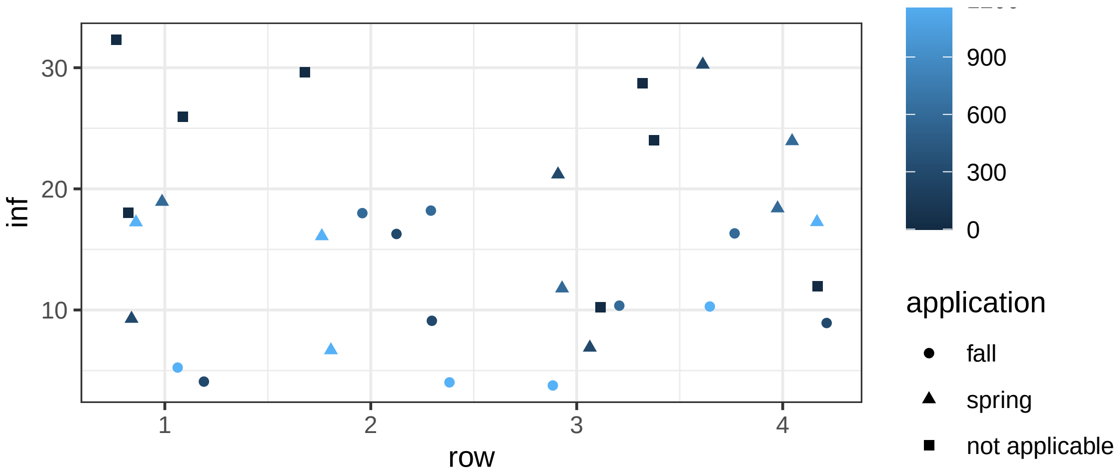
Data


Average scab percent
30
20
10
0
250
500
750
1000
1250
Sulfur
application
fall
spring
not applicable

Data


30
20
10
inf
2
4
6
8
col
900
600
300
0
application
fall
spring
not applicable

Data


inf
30
20
10
1
2
3
4
row
900
600
300
0
application
fall
spring
not applicable

Model

Y_{ij}: avg % of surface area covered with scab for plot i in treatment j for $j = 1, \ldots, 7$.

Assume $Y_{ij} \stackrel{ind}{\sim} N(\mu_j, \sigma^2)$.

Hypotheses:

- Difference amongst any means:
  One-way ANOVA F-test
- *Any effect*:
  Contrast: control vs sulfur
- *Fall vs spring*:
  Contrast: fall vs spring applications
- *Sulfur level*:
  Contrast: linear trend

Contrasts

- *Sulfur effect*: Any sulfur vs none

$$
\begin{aligned}
\gamma &= \tfrac{1}{6}(\mu_{F12} + \mu_{F6} + \mu_{F3} + \mu_{S3} + \mu_{S6} + \mu_{S12}) - \mu_O \\
&= \tfrac{1}{6}(\mu_{F12} + \mu_{F6} + \mu_{F3} + \mu_{S3} + \mu_{S6} + \mu_{S12} - 6\mu_O)
\end{aligned}
$$

- *Fall vs spring*: Contrast comparing fall vs spring applications

$$
\begin{aligned}
\gamma &= \tfrac{1}{3}(\mu_{F12} + \mu_{F6} + \mu_{F3}) + 0\mu_O - \tfrac{1}{3}(\mu_{S3} + \mu_{S6} + \mu_{S12}) \\
&= \tfrac{1}{3}\left[1\mu_{F12} + 1\mu_{F6} + 1\mu_{F3} + 0\mu_O - 1\mu_{S3} - 1\mu_{S6} - 1\mu_{S12}\right]
\end{aligned}
$$

Contrasts (cont.)

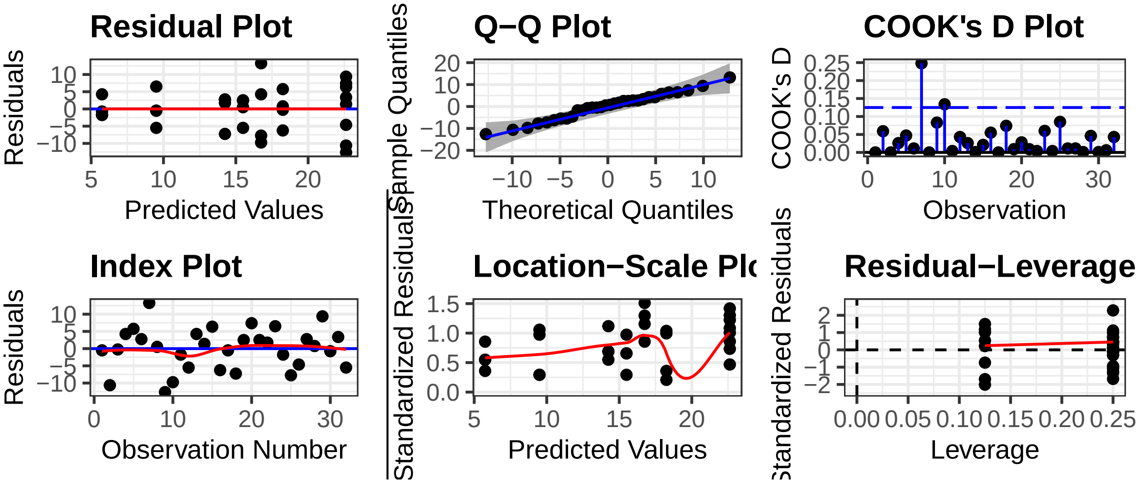
- Sulfur linear trend
  - The group sulfur levels (X_j) are 12, 6, 3, 0, 3, 6, and 12 (100 lbs/acre)
  - and a linear trend contrast is $X_j - \overline{X}$

X_i	12	6	3	0	3	6	12
$X_i - \overline{X}$	6	0	-3	-6	-3	0	6

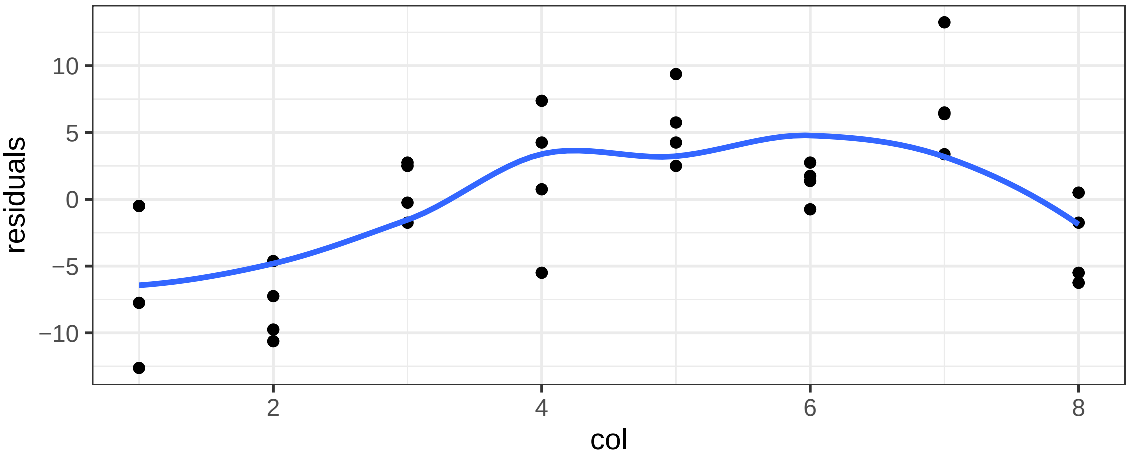
$$\gamma \quad = 6\mu_{F12} + 0\mu_{F6} - 3\mu_{F3} - 6\mu_O - 3\mu_{S3} + 0\mu_{S6} + 6\mu_{S12}$$

Trt	F12	F6	F3	O	S3	S6	S12	Div
Sulfur v control	1	1	1	-6	1	1	1	6
Fall v Spring	1	1	1	0	-1	-1	-1	3
Linear Trend	6	0	-3	-6	-3	0	6	1

```r
K <-
#                                       F12 F6 F3  0 S3 S6 S12
               list("sulfur - control" = c( 1, 1, 1,-6, 1, 1,  1)/6,
                    "fall - spring"    = c( 1, 1, 1, 0,-1,-1, -1)/3,
                    "linear trend"     = c( 6, 0,-3,-6,-3, 0,  6)/1)

m <- lm(inf ~ treatment, data = d)
anova(m)

Analysis of Variance Table

Response: inf
          Df  Sum Sq Mean Sq F value  Pr(>F)
treatment  6  972.34 162.057  3.6081 0.01026 *
Residuals 25 1122.88  44.915
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```


Residual Plot
Residuals
Predicted Values
Q−Q Plot
Sample Quantiles
Theoretical Quantiles
COOK's D Plot
COOK's D
Observation
Index Plot
Residuals
Observation Number
Location−Scale Plo
Standardized Residuals
Predicted Values
Residual−Leverage
Standardized Residuals
Leverage

```
em <- emmeans(m, ~treatment); em

 treatment emmean   SE df lower.CL upper.CL
 F12         5.75 3.35 25    -1.15     12.7
 F6         15.50 3.35 25     8.60     22.4
 F3          9.50 3.35 25     2.60     16.4
 O          22.62 2.37 25    17.74     27.5
 S3         16.75 3.35 25     9.85     23.7
 S6         18.25 3.35 25    11.35     25.2
 S12        14.25 3.35 25     7.35     21.2

Confidence level used: 0.95

co <- contrast(em, K)
confint(co)

 contrast         estimate    SE df lower.CL upper.CL
 sulfur - control    -9.29  2.74 25    -14.9   -3.657
 fall - spring       -6.17  2.74 25    -11.8   -0.532
 linear trend       -94.50 34.82 25   -166.2  -22.779

Confidence level used: 0.95
```


10
5
0
−5
−10
residuals
2
4
6
8
col

Summary

For this particular data analysis

- Significant differences in means between the groups (ANOVA $F_{6,25} = 3.61$ p=0.01)
- Having sulfur was associated with a reduction in scab % of 9 (4,15) compared to no sulfur
- Fall application reduced scab % by 6 (0.5,12) compared to spring application
- Linear trend in sulfur was significant (p=0.01)

- Concerned about spatial correlation among columns
- Consider a logarithm of the response
  - CI for F12 (-1.2, 12.7)
  - Non-constant variance (residuals vs predicted, sulfur, application)