

# S01 - Logistic Regression

STAT 401 (Engineering) - Iowa State University

April 23, 2018

# Linear regression

The linear regression model

$$Y_i \overset{ind}{\sim} N(\mu_i, \sigma^2)$$
$$\mu_i = \beta_0 + \beta_1 X_{i,1} + \cdots + \beta_p X_{i,p}$$

where

- $Y_i$  is continuous
- $X_i$  is continuous or categorical (indicator variables)

What if  $Y_i$  is

- binary or
- a count (of the number of success out of some total)?

# Logistic regression

Let

$$Y_i = \begin{cases} 1 & \text{if observation } i \text{ is a "success"} \\ 0 & \text{otherwise.} \end{cases}$$

and  $X_i$  be an explanatory variable that affects the probability of success  $\theta_i$  for observation  $i$ .

Then a logistic regression model is

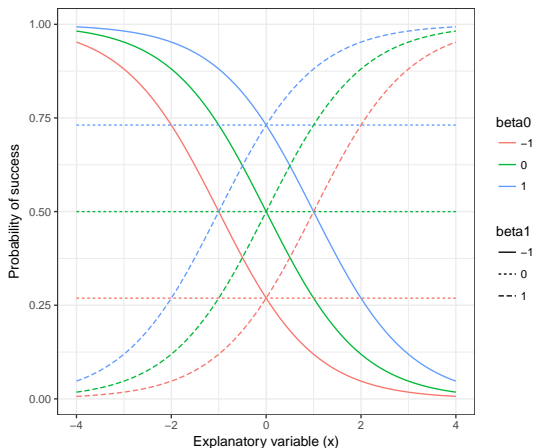
$$Y_i \stackrel{ind}{\sim} \text{Ber}(\theta_i) \quad \text{and} \quad \text{logit}(\theta_i) = \log\left(\frac{\theta_i}{1 - \theta_i}\right) = \beta_0 + \beta_1 X_i$$

where the logistic function of  $X_i$  is

$$\theta_i = f(X_i) = \frac{e^{\beta_0 + \beta_1 X_i}}{1 + e^{\beta_0 + \beta_1 X_i}} = \frac{1}{1 + e^{-(\beta_0 + \beta_1 X_i)}}.$$

```
d <- expand.grid(b0 = c(-1,0,1), b1 = c(-1,0,1), x = seq(-4,4,by=0.1)) %>%
  mutate(theta = 1/(1+exp(-(b0+b1*x))),
         beta0 = as.factor(b0),
         beta1 = as.factor(b1))

ggplot(d, aes(x,theta,color=beta0,linetype=beta1,group=interaction(beta0,beta1))) +
  geom_line() +
  theme_bw() +
  labs(x="Explanatory variable (x)", y="Probability of success")
```



# Interpretation

When  $X_i = 0$ , then

$$E[Y_i|X_i = 0] = \theta_i = \frac{1}{1 + e^{-\beta_0}}$$

thus  $\beta_0$  determines the **probability of success when the explanatory variable is zero**.

The odds of success when  $X_1 = x$  is

$$\frac{\theta_1}{1 - \theta_1} = e^{\beta_0 + \beta_1 x}.$$

The probability of success when  $X_2 = x + 1$  is

$$\frac{\theta_2}{1 - \theta_2} = e^{\beta_0 + \beta_1(x+1)} = e^{\beta_0 + \beta_1 x + \beta_1}.$$

Thus, the multiplicative change in the odds for a 1 unit increase in  $x$  is

$$\frac{\frac{\theta_2}{1 - \theta_2}}{\frac{\theta_1}{1 - \theta_1}} = \frac{e^{\beta_0 + \beta_1 x + \beta_1}}{e^{\beta_0 + \beta_1 x}} = e^{\beta_1}$$

This is also referred to as an **odds ratio**.

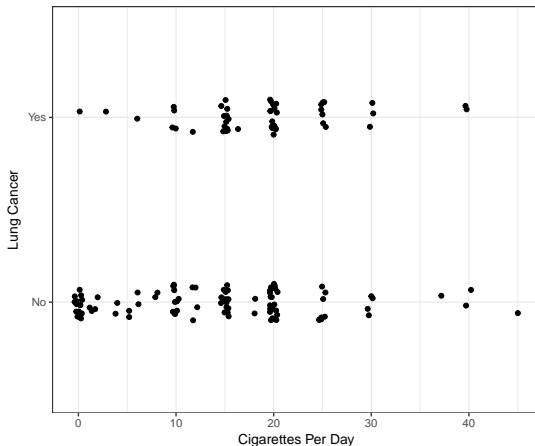
# Lung cancer due to smoking

To *prove* a causal relationship between lung cancer and smoking, there should be clear evidence that there is a dose response between lung cancer and smoking.

But since lung cancer is binary, we need to compare the proportion of individuals who have lung cancer to those who don't amongst the individuals who smoke about the same amount.

To investigate the causes of lung cancer, researchers conducted a **case-control** study where the 49 cases of individuals with lung cancer were *matched* with 98 controls from a population of residents having the same general age structure. (In case-control studies, the intercept does not have our standard interpretation because it is determined by our sampling.)

```
lung_cancer <- Sleuth3::case2002 %>%  
  mutate(`Lung Cancer` = ifelse(LC=="NoCancer", "No", "Yes"),  
         `Cigarettes Per Day` = CD)  
  
ggplot(lung_cancer, aes(`Cigarettes Per Day`, `Lung Cancer`)) +  
  geom_jitter(height=0.1) +  
  theme_bw()
```



# Analysis

```
m <- glm(`Lung Cancer`=="Yes" ~ `Cigarettes Per Day`,
  data = lung_cancer,
  family = "binomial")

summary(m)
```

Call:

```
glm(formula = `Lung Cancer` == "Yes" ~ `Cigarettes Per Day`,
  family = "binomial", data = lung_cancer)
```

Deviance Residuals:

Min	1Q	Median	3Q	Max
-1.5148	-0.9688	-0.7166	1.3449	1.8603

Coefficients:

	Estimate	Std. Error	z value	Pr(> z )
(Intercept)	-1.53541	0.37707	-4.072	4.66e-05 ***
`Cigarettes Per Day`	0.05113	0.01939	2.637	0.00836 **

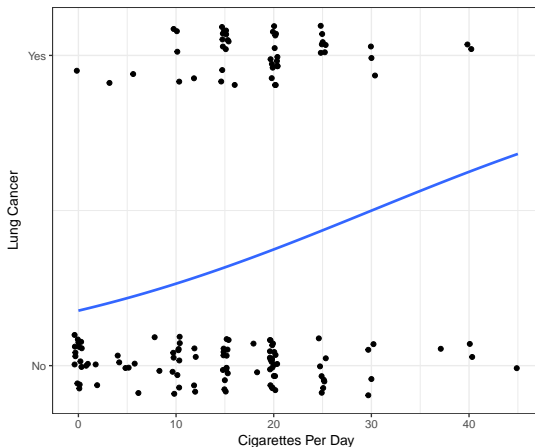
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 Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for binomial family taken to be 1)

Null deviance:	187.14	on 146	degrees of freedom
Residual deviance:	179.62	on 145	degrees of freedom
AIC:	183.62		

Number of Fisher Scoring iterations: 4

```
ggplot(lung_cancer, aes(`Cigarettes Per Day`, 1*(`Lung Cancer` == "Yes"))) +  
  geom_jitter(height=0.1) +  
  stat_smooth(method="glm",  
             se=FALSE,  
             method.args = list(family="binomial")) +  
  theme_bw() +  
  scale_y_continuous(breaks=c(0,1), labels=c("No", "Yes")) +  
  labs(y = "Lung Cancer")
```



# Grouping

Often data are grouped:

```
lung_cancer_grouped <- lung_cancer %>%
  group_by(`Cigarettes Per Day`) %>%
  summarize(`Number of individuals` = n(),
            `Number with lung cancer` = sum(`Lung Cancer` == "Yes"),
            `Number without lung cancer` = sum(`Lung Cancer` == "No"),
            `Proportion with lung cancer` = `Number with lung cancer` / `Number of individuals`)
```

lung\_cancer\_grouped

# A tibble: 19 x 5

	`Cigarettes Per Day` <int>	`Number of individuals` <int>	`Number with lung cancer` <int>	`Number without lung cancer` <int>	`Proportion with lung cancer` <dbl>
1	0	17	1	16	0.0588
2	1	2	0	2	0.0000
3	2	2	0	2	0.0000
4	3	1	1	0	1.0000
5	4	2	0	2	0.0000
6	5	2	0	2	0.0000
7	6	3	1	2	0.3333
8	8	2	0	2	0.0000
9	10	15	4	11	0.2667
10	12	5	1	4	0.2000
11	15	27	12	15	0.4444
12	16	1	1	0	1.0000
13	18	2	0	2	0.0000
14	20	38	16	22	0.4211
15	25	15	7	8	0.4667
16	30	7	3	4	0.7500
17	37	1	0	1	0.0000

# Binomial distribution

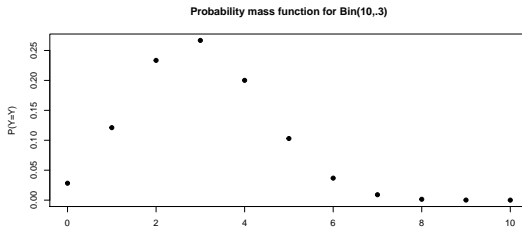
The probability mass function of the binomial distribution is

$$P(Y = y) = \binom{n}{y} \theta^y (1 - \theta)^{n-y} \quad y = 0, 1, 2, \dots, n$$

Properties:

- $E[Y] = n\theta$
- $V[Y] = n\theta(1 - \theta)$

```
xx = 0:10  
plot(xx, dbinom(xx, 10, .3), main="Probability mass function for Bin(10,.3)",  
      xlab="y", ylab="P(Y=Y)", pch=19)
```



# Logistic regression for grouped data

Let  $Y_i$  be the number of success out of  $n_i$  attempts in group  $i$ . Then a logistic regression model is

$$Y_i \overset{ind}{\sim} \text{Bin}(n_i, \theta_i)$$

$$\text{logit}(\theta_i) = \log\left(\frac{\theta_i}{1-\theta_i}\right) = \beta_0 + \beta_1 X_i$$

where

- $Y_i$  is an integer from 0 to  $n_i$
- Bin refers to the binomial distribution

# Logistic regression in R

```
m = glm(cbind(`Number with lung cancer`, `Number without lung cancer`) ~ `Cigarettes Per Day`,
        data = lung_cancer_grouped,
        family="binomial")
summary(m)
```

```
Call:
glm(formula = cbind(`Number with lung cancer`, `Number without lung cancer`) ~
    `Cigarettes Per Day`, family = "binomial", data = lung_cancer_grouped)
```

Deviance Residuals:

Min	1Q	Median	3Q	Max
-1.5148	-1.0253	-0.5070	0.3305	1.7922

Coefficients:

	Estimate	Std. Error	z value	Pr(> z )
(Intercept)	-1.53541	0.37707	-4.072	4.66e-05 ***
`Cigarettes Per Day`	0.05113	0.01939	2.637	0.00836 **

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Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for binomial family taken to be 1)

Null deviance: 28.651 on 18 degrees of freedom  
 Residual deviance: 21.141 on 17 degrees of freedom  
 AIC: 48.879

Number of Fisher Scoring iterations: 4

```
confint(m)
```

## Effect of birdkeeping on lung cancer

The data set we have been analyzing was actually constructed to investigate the relationship between birdkeeping and lung cancer. But, since we know smoking increase the probability of developing lung cancer, we want to **control** for the effect of smoking when assessing the effect of bird keeping. Thus, we will run a logistic regression with both smoking and bird-keeping to determine the effect of bird-keeping on lung cancer.

# Summarize data

```
lung_cancer_bird <- Sleuth3::case2002 %>%
  group_by(CD, BK) %>%
  summarize(y = sum(LC == "LungCancer"),
            n = n(),
            p = y/n)
```

```
lung_cancer_bird
```

```
# A tibble: 30 x 5
```

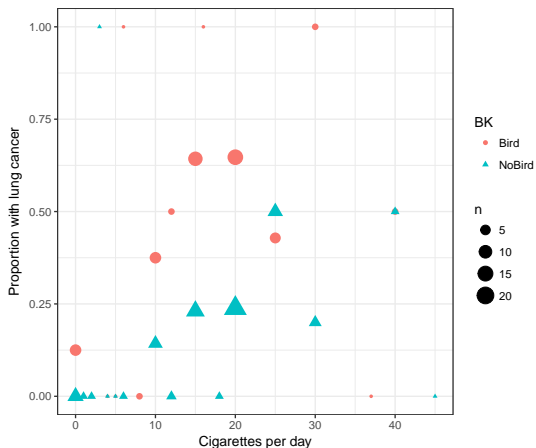
```
# Groups:   CD [?]
```

	CD	BK	y	n	p
	<int>	<fct>	<int>	<int>	<dbl>
1	0	Bird	1	8	0.125
2	0	NoBird	0	9	0
3	1	NoBird	0	2	0
4	2	NoBird	0	2	0
5	3	NoBird	1	1	1.00
6	4	Bird	0	1	0
7	4	NoBird	0	1	0
8	5	Bird	0	1	0
9	5	NoBird	0	1	0
10	6	Bird	1	1	1.00

```
# ... with 20 more rows
```

# Visualize data

```
ggplot(lung_cancer_bird, aes(CD, p, size=n, color=BK, shape=BK)) +  
  geom_point() +  
  theme_bw() +  
  labs(x="Cigarettes per day", y="Proportion with lung cancer")
```



# Model

Let  $Y_i$  be the number of success out of  $n_i$  attempts in group  $i$  with explanatory variables  $X_{i,1}$  and  $X_{i,2}$ . Then a logistic regression model is

$$Y_i \stackrel{ind}{\sim} \text{Bin}(n_i, \theta_i)$$

$$\text{logit}(\theta_i) = \log\left(\frac{\theta_i}{1-\theta_i}\right) = \beta_0 + \beta_1 X_{i,1} + \beta_2 X_{i,2}$$

The interpretation is

- The probability of success is  $\frac{1}{1+e^{-\beta_0}}$  when all explanatory variables are zero. (Except in a case-control study.)
- The odds ratio for a one unit increase in  $X_{i,1}$  is  $e^{\beta_1}$  when holding all other explanatory variables constant.
- The odds ratio for a one unit increase in  $X_{i,2}$  is  $e^{\beta_2}$  when holding all other explanatory variables constant.

# Logistic regression with multiple explanatory variables

```
# LC is binary
summary(m <- glm(cbind(y,n-y) ~ CD + BK, data=lung_cancer_bird, family="binomial"))
```

Call:

```
glm(formula = cbind(y, n - y) ~ CD + BK, family = "binomial",
    data = lung_cancer_bird)
```

Deviance Residuals:

	Min	1Q	Median	3Q	Max
	-1.7167	-0.9555	-0.5413	0.4025	2.1594

Coefficients:

	Estimate	Std. Error	z value	Pr(> z )
(Intercept)	-0.94683	0.41319	-2.291	0.021935 *
CD	0.05838	0.02087	2.797	0.005157 **
BKNoBird	-1.45760	0.38856	-3.751	0.000176 ***

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Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for binomial family taken to be 1)

Null deviance: 53.386 on 29 degrees of freedom  
 Residual deviance: 30.612 on 27 degrees of freedom  
 AIC: 66.07

Number of Fisher Scoring iterations: 4

```
nd <- expand.grid(CD = 0:45, BK=c("Bird","NoBird"))
pd <- cbind(nd, data.frame(p=predict(m, newdata = nd, type = "response")))

ggplot() +
  geom_point(data = lung_cancer_bird, aes(CD, p, size=n, color=BK, shape=BK)) +
  geom_line(data = pd, aes(CD, p, color=BK, linetype=BK)) +
  theme_bw() +
  labs(x="Cigarettes per day", y="Proportion with lung cancer")
```

