Polynomial Trend Models

Dr. Jarad Niemi

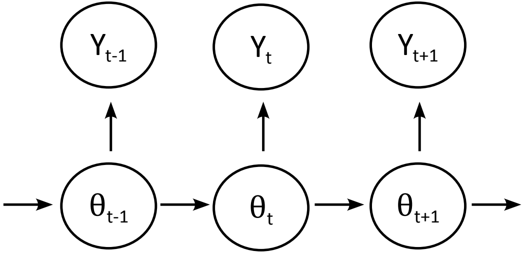
STAT 6150 - Iowa State University

October 8, 2025

Dynamic Linear Models

$$\begin{aligned}
Y_t &= F_t\theta_t + v_t & v_t &\stackrel{ind}{\sim} N_m(0, V_t) \\
\theta_t &= G_t\theta_{t-1} + w_t & w_t &\stackrel{ind}{\sim} N_p(0, W_t) \\
& & \theta_0 &\sim N_p(m_0, C_0)
\end{aligned}$$

where v_t and w_t are independent across time and all are independent of θ_0.



Decomposition of time series

Consider a univariate series Y_t. Think of this series has being the sum of independent components

$$Y_t = Y_{1,t} + \cdots + Y_{h,t} = \sum_{i=1}^{h} Y_{i,t}$$

where each component has its own independent DLM (dynamic linear model),

$$\begin{aligned} Y_{i,t} &= F_{i,t}\theta_{i,t} + v_{i,t} & v_{i,t} &\sim N(0, V_{i,t}) \\ \theta_{i,t} &= G_{i,t}\theta_{i,t-1} + w_{i,t} & w_{i,t} &\sim N(0, W_{i,t}) \end{aligned}$$

Then

$$\begin{aligned} Y_t &= F_t\theta_t + v_t & v_t &\sim N(0, V_t) \\ \theta_t &= G_t\theta_{t-1} + w_t & w_t &\sim N(0, W_t) \end{aligned}$$

Observation equation

Recall

$$
\begin{aligned}
Y_t &= Y_{1,t} + \cdots + Y_{h,t} \\
Y_{i,t} &= F_{i,t}\theta_{i,t} + v_{i,t} \qquad\qquad v_{i,t} \sim N(0, V_{i,t})
\end{aligned}
$$

$$
\begin{aligned}
Y_t &= \sum_{i=1}^h Y_{i,t} \\
&= \sum_{i=1}^h \left[F_{i,t}\theta_{i,t} + v_{i,t}\right] \\
&= \sum_{i=1}^h F_{i,t}\theta_{i,t} + \sum_{i=1}^h v_{i,t} \\
&= \sum_{i=1}^h F_{i,t}\theta_{i,t} + v_t \qquad\qquad v_t \sim N(0, V_t) \quad V_t = \sum_{i=1}^h V_{i,t} \\
&= F_t\theta_t + v_t
\end{aligned}
$$

where

$$
\theta_t = \left[\begin{array}{c} \theta_{1,t} \\ \vdots \\ \theta_{h,t} \end{array}\right] \qquad\qquad F_t = [F_{1,t}|\cdots|F_{h,t}]
$$

Evolution equation

Recall

$$\theta_t = \begin{bmatrix} \theta_{1,t} \\ \vdots \\ \theta_{h,t} \end{bmatrix} = \begin{bmatrix} G_{1,t}\theta_{1,t-1} + w_{1,t} \\ \vdots \\ G_{h,t}\theta_{h,t-1} + w_{h,t} \end{bmatrix} = \begin{bmatrix} G_{1,t}\theta_{1,t-1} \\ \vdots \\ G_{h,t}\theta_{h,t-1} \end{bmatrix} + \begin{bmatrix} w_{1,t} \\ \vdots \\ w_{h,t} \end{bmatrix}$$

$$= \begin{bmatrix} G_{1,t}\theta_{1,t-1} \\ \vdots \\ G_{h,t}\theta_{h,t-1} \end{bmatrix} + w_t = G_t\theta_{t-1} + w_t$$

where $w_t \stackrel{ind}{\sim} N(0, W_t)$ and

$$W_t = \begin{bmatrix} W_{1,t} & & \\ & \ddots & \\ & & W_{h,t} \end{bmatrix} \qquad G_t = \begin{bmatrix} G_{1,t} & & \\ & \ddots & \\ & & G_{h,t} \end{bmatrix}$$

Combining model components

Consider

$$\begin{aligned} Y_t &= F_t\theta_t + v_t & v_t &\stackrel{ind}{\sim} N(0, V_t) \\ \theta_t &= G_t\theta_{t-1} + w_t & w_t &\stackrel{ind}{\sim} N(0, W_t) \end{aligned}$$

where $V_t = \sum_{i=1}^h V_{i,t}$,

$$\theta_t = \left[\begin{array}{c} \theta_{1,t} \\ \vdots \\ \theta_{h,t} \end{array}\right] \qquad F_t = [F_{1,t}|\cdots|F_{h,t}]$$

and

$$W_t = \left[\begin{array}{ccc} W_{1,t} & & \\ & \ddots & \\ & & W_{h,t} \end{array}\right] \qquad G_t = \left[\begin{array}{ccc} G_{1,t} & & \\ & \ddots & \\ & & G_{h,t} \end{array}\right]$$

Polynomial trend model definition

A polynomial model of order n is a DLM with constant [and specified] matrices $F_t = F$ and $G_t = G$, and a forecast function of the form

$$f_t(k) = E(Y_{t+k}|y_{1:t}) = a_{t,0} + a_{t,1}k + \cdots + a_{t,n-1}k^{n-1},\ k \geq 0$$

where $a_{t,0}, \ldots, a_{t,n-1}$ are linear functions of $m_t = E(\theta_t|y_{1:t})$ and are independent of k. Thus, the forecast function is a polynomial or order $n-1$ in k.

In practice we use,

- Local level model ($n = 1$).
- Linear trend model ($n = 2$).
- Exponential trends are accommodated by taking logs and then using a linear trend model.

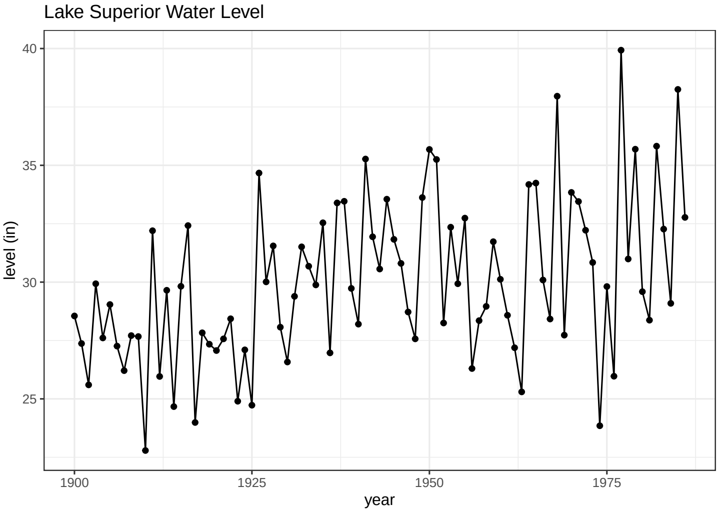
Local level model

$$
\begin{aligned}
Y_t &= \theta_t + v_t & v_t &\stackrel{ind}{\sim} N(0, V) \\
\theta_t &= \theta_{t-1} + w_t & w_t &\stackrel{ind}{\sim} N(0, W) \\
& & \theta_0 &\sim N_p(m_0, C_0)
\end{aligned}
$$

where $F_t = F = 1$, $G_t = G = 1$, $V_t = V$, and $W_t = W$.

What is the forecast function? $f_t(k) = E(Y_{t+k}|y_{1:t}) = m_t$.

Lake Superior Data

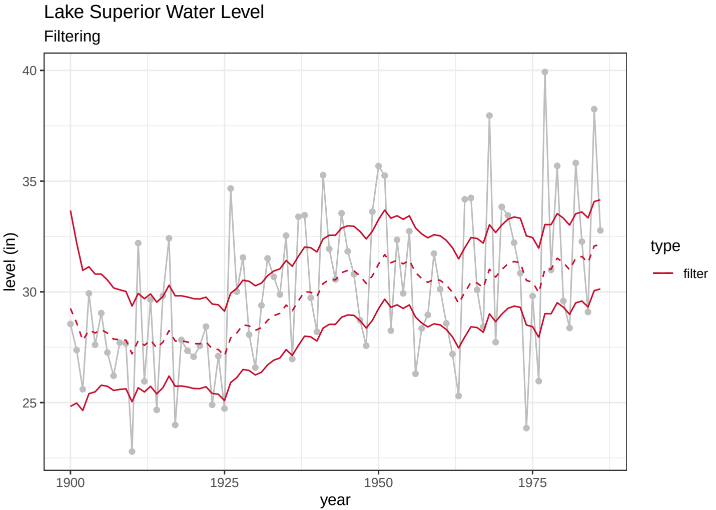


Fit a local level DLM

```r
# Obtain MLEs for V and W
build_ll_model <- function(theta, ...) {
  dlmModPoly(order = 1,
             dV = exp(theta[1]),
             dW = exp(theta[2]),
             ...)
}

lakeSup_MLEs <- dlmMLE(lakeSup$`level (in)`, rep(0, 2), build_ll_model)
stopifnot(!lakeSup_MLEs$convergence) # 0 is converged

exp(lakeSup_MLEs$par) # estimated V and W

## [1] 9.4654447 0.1211534

# Estimate state (assuming these values for V and W)
model_local_level <- build_ll_model(lakeSup_MLEs$par, m0 = 30, C0 = 10)

lSupFilt <- dlmFilter(lakeSup$`level (in)`, model_local_level)
lSupSmoo <- dlmSmooth(lSupFilt)
lSupFore <- dlmForecast(lSupFilt, nAhead = 10)
```

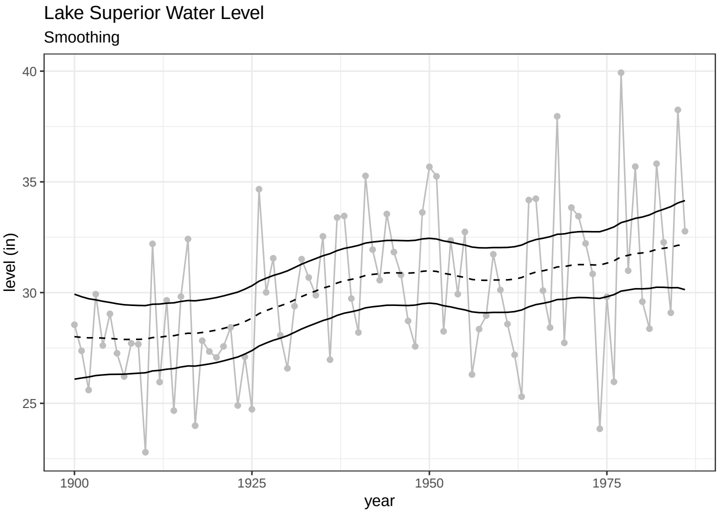
```r
# Construct data.frame for plotting
lakeSup_plot <-
  bind_rows(
    lakeSup |>
      mutate(
        type = "filter",
        mean = lSupFilt$m[-1],
        sd   = sqrt(unlist(dlmSvd2var(lSupFilt$U.C, lSupFilt$D.C)))[-1]),

    lakeSup |>
      mutate(
        type = "smooth",
        mean = lSupSmoo$s[-1],
        sd   = sqrt(unlist(dlmSvd2var(lSupSmoo$U.S, lSupSmoo$D.S)))[-1]),

    data.frame(
      type = "forecast-state",
      year = max(lakeSup$year) + 1:nrow(lSupFore$a),
      mean = as.numeric(lSupFore$a),
      sd   = sqrt(unlist(lSupFore$R))
    ),

    data.frame(
      type = "forecast-observation",
      year = max(lakeSup$year) + 1:nrow(lSupFore$a),
      mean = as.numeric(lSupFore$f),
      sd   = sqrt(unlist(lSupFore$Q))
    )
  ) |>
  mutate(
    lb = mean - 2 * sd,
    ub = mean + 2 * sd
  )
```

Lake Superior - Local Level Model

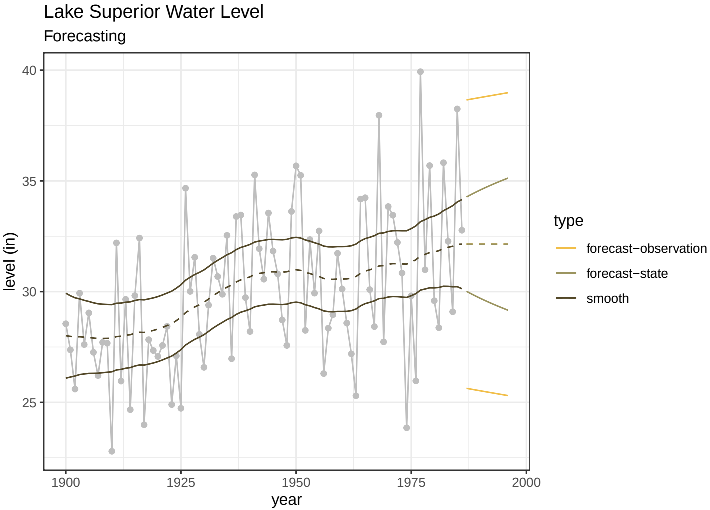


Lake Superior - Local Level Model

```
## Warning:  No shared levels found between `names(values)` of the manual scale and the data's colour
values.
```



Lake Superior - Local Level Model



Linear trend model

$$\begin{aligned} Y_t &= \mu_t + v_t & v_t &\stackrel{ind}{\sim} N(0,V) \\ \mu_t &= \mu_{t-1} + \beta_{t-1} + w_{t,1} & w_{t,1} &\stackrel{ind}{\sim} N(0,\sigma_\mu^2) \\ \beta_t &= \beta_{t-1} + w_{t,2} & w_{t,2} &\stackrel{ind}{\sim} N(0,\sigma_\beta^2) \\ & & \theta_0 &\sim N_p(m_0, C_0) \end{aligned}$$

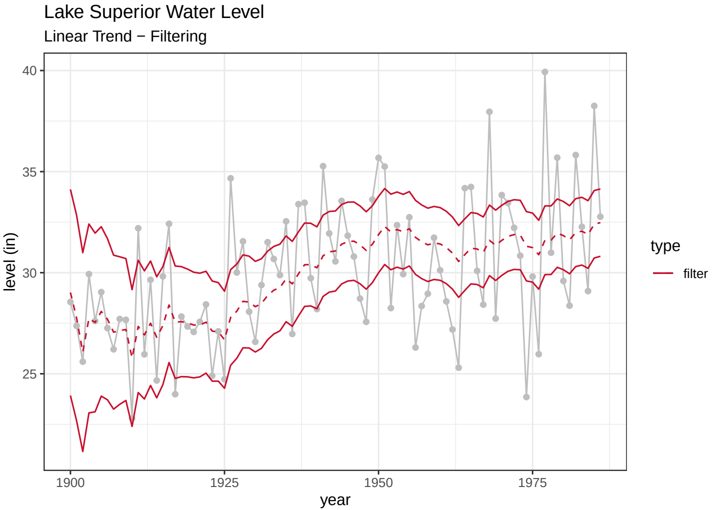
- $F_t = F = (1,0)$
- $\theta_t = (\mu_t, \beta_t)^\top$
- $$G_t = \left[\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right]$$
- Forecast function $f_t(k) = E[Y_{t+k}|y_{1:t}] = \hat{\mu}_t + k\hat{\beta}_t$

Fit a local level DLM

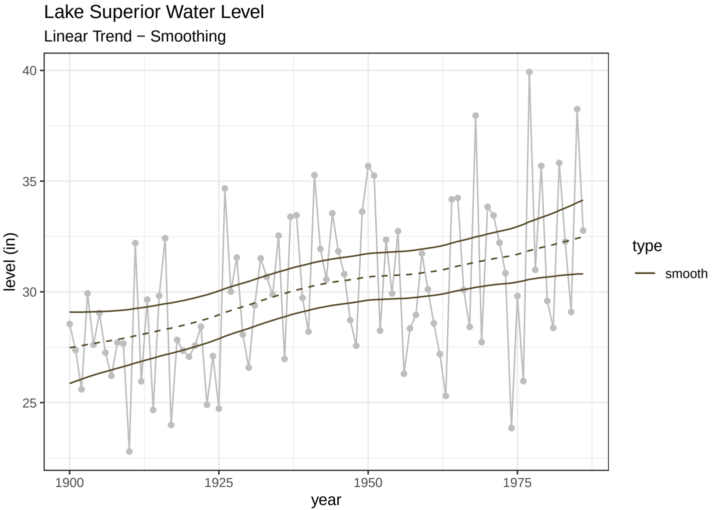
```r
# Obtain MLEs for V, W[1,1], and W[2,2]
build_lt_model <- function(theta, ...) {
  dlmModPoly(order = 2,
             dV = exp(theta[1]),
             dW = exp(theta[2:3]),
             ...)
}

lakeSup_lt_MLEs <- dlmMLE(lakeSup$`level (in)`, rep(0, 3), build_lt_model)
stopifnot(!lakeSup_lt_MLEs$convergence) # 0 is converged

exp(lakeSup_lt_MLEs$par) # estimated V, W[1,1], and W[2,2]

## [1] 9.623499e+00 3.039463e-02 8.014774e-08

# Estimate state (assuming these values for V and W)
model_linear_trend <- build_lt_model(lakeSup_lt_MLEs$par, m0 = c(30,0), C0 = diag(10, 2, 2))

ltSupFilt <- dlmFilter(lakeSup$`level (in)`, model_linear_trend)
ltSupSmoo <- dlmSmooth(ltSupFilt)
ltSupFore <- dlmForecast(ltSupFilt, nAhead = 10)
```

```r
# Construct data.frame for plotting
lakeSup_lt_plot <-
  bind_rows(
    lakeSup |>
      mutate(
        type = "filter",
        mean = ltSupFilt$m[-1,1],
        sd   = sqrt(unlist(lapply(dlmSvd2var(ltSupFilt$U.C,
                                             ltSupFilt$D.C),
                                 function(x) x[1]))[-1])),

    lakeSup |>
      mutate(
        type = "smooth",
        mean = ltSupSmoo$s[-1,1],
        sd   = sqrt(unlist(lapply(dlmSvd2var(ltSupSmoo$U.S,
                                             ltSupSmoo$D.S),
                                 function(x) x[1]))[-1])),

    data.frame(
      type = "forecast-state",
      year = max(lakeSup$year) + 1:nrow(ltSupFore$a),
      mean = as.numeric(ltSupFore$a[,1]),
      sd   = sqrt(unlist(lapply(ltSupFore$R, function(x) x[1])))
    ),

    data.frame(
      type = "forecast-observation",
      year = max(lakeSup$year) + 1:nrow(ltSupFore$a),
      mean = as.numeric(ltSupFore$f[,1]),
      sd   = sqrt(unlist(lapply(ltSupFore$Q, function(x) x[1])))
    )
  ) |>
  mutate(lb = mean - 2 * sd, ub = mean + 2 * sd)
```

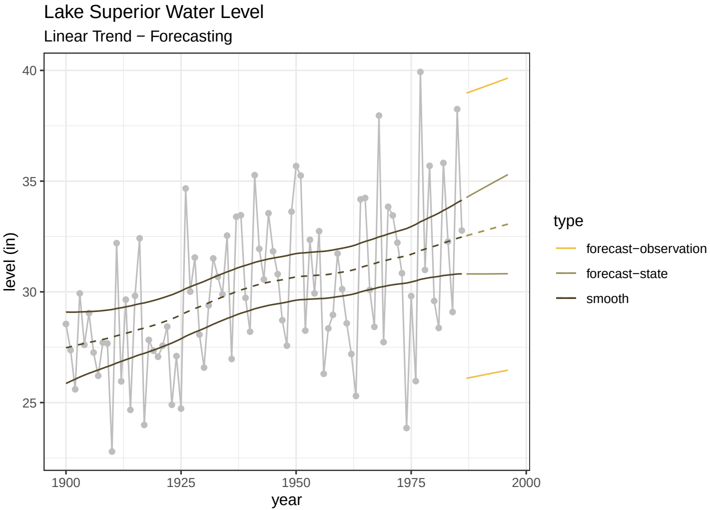
Lake Superior - Linear Trend Model



Lake Superior - Linear Trend Model



Lake Superior - Linear Trend Model



Polynomial trend models

$$
\begin{aligned}
Y_t &= F_t\theta_t + v_t & v_t &\stackrel{ind}{\sim} N_m(0, V_t) \\
\theta_t &= G_t\theta_{t-1} + w_t & w_t &\stackrel{ind}{\sim} N_p(0, W_t) \\
& & \theta_0 &\sim N_p(m_0, C_0)
\end{aligned}
$$

- $F_t = F = (1, 0, \ldots, 0)$
- $$G_t = G = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & & \cdots & 0 & 1 & 1 \\ 0 & & \cdots & & 0 & 1 \end{bmatrix}$$
- $W_t = W = \text{diag}(W_1, \ldots, W_n)$